

Diophantine tori and Weyl laws for non-selfadjoint operators in dimension two

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Abstract: We study the distribution of eigenvalues for non-selfadjoint perturbations of selfadjoint semiclassical analytic pseudodifferential operators in dimension two, assuming that the classical flow of the unperturbed part is completely integrable. An asymptotic formula of Weyl type for the number of eigenvalues in a spectral band, bounded from above and from below by levels corresponding to Diophantine invariant Lagrangian tori, is established. The Weyl law is given in terms of the long time averages of the leading non-selfadjoint perturbation along the classical flow of the unperturbed part.

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1 Introduction

Among the various results in spectral analysis for selfadjoint linear partial differential operators, a distinguished role is played by the Weyl law, remarkable for its generality and simplicity— see [6], [18] for a detailed discussion and numerous references. Let us recall here a rough statement of it in the semiclassical case. Let $P = p^w(x, hD_x)$, $0 < h \leq 1$, be the semiclassical Weyl quantization on \mathbf{R}^n of a real-valued smooth symbol p belonging to a suitable symbol class and satisfying an ellipticity condition at infinity, guaranteeing that the spectrum of P is discrete in a fixed open set $\Omega \subset \mathbf{R}$. When $[E_1, E_2] \subset \Omega$ is an interval such that $\text{vol}(p^{-1}(E_j)) = 0$, $j = 1, 2$, then the number $N(P, [E_1, E_2])$ of eigenvalues of P in $[E_1, E_2]$ satisfies

$$N(P, [E_1, E_2]) = \frac{1}{(2\pi h)^n} \left(\text{vol}(p^{-1}([E_1, E_2])) + o(1) \right), \quad h \rightarrow 0. \quad (1.1)$$

The leading coefficient in the Weyl law (1.1), given by the phase space volume corresponding to the energy range $[E_1, E_2]$, captures the basic physical intuition of each quantum state occupying a fixed volume $(2\pi h)^n$ in the phase space. We remark that the corresponding development for selfadjoint partial differential operators in the high energy limit has a long and distinguished tradition, starting with the works of H. Weyl, see [41].

The situation becomes quite different in the non-selfadjoint analytic case, where the asymptotics of the counting function for eigenvalues may no longer be governed by volumes of subsets of the real phase space, and the spectrum is often determined by the behavior of the holomorphic continuation of the symbol along suitable complex deformations of the real phase space. Following [3], [10], we may consider, for instance, the complex harmonic oscillator $P = 1/2((hD_x)^2 + ix^2)$ on $L^2(\mathbf{R})$. Here the range of the symbol on \mathbf{R}^2 is the closed first quadrant, while according to the general results of [28], the spectrum of P is equal to $\{e^{i\pi/4}(k + \frac{1}{2})h; k \in \mathbf{N}\}$. In this case, it turns out that the natural phase space associated to P is given by $T^*(e^{-i\frac{\pi}{8}}\mathbf{R})$ —

see also [11] for the precise spectral results in dimension one, closely related to this example.

In the paper [25], it has been shown that for large and stable classes of non-selfadjoint analytic operators in dimension two, the individual eigenvalues can be determined accurately in the semiclassical limit by means of a Bohr-Sommerfeld quantization condition, defined in terms of suitable complex Lagrangian tori close to the real domain. (See also [35] for the formulation of the corresponding Weyl laws.) The work [25] was subsequently continued in a series of papers [12]–[14], [16], [15], all of them concerned with the case of non-selfadjoint perturbations of selfadjoint operators of the form,

$$P_\varepsilon(x, hD_x) = p^w(x, hD_x) + i\varepsilon q^w(x, hD_x), \quad 0 < \varepsilon \ll 1,$$

with the leading symbol $p_\varepsilon(x, \xi) = p(x, \xi) + i\varepsilon q(x, \xi)$, $(x, \xi) \in T^*\mathbf{R}^2$. Here p is real, so that $P_{\varepsilon=0}$ is selfadjoint, and we assume that both p and q are analytic, with p elliptic outside a compact set. The spectrum of P_ε near the origin is confined to a band of height $\mathcal{O}(\varepsilon)$, and to study the imaginary parts of the eigenvalues in the band, following the classical averaging method [40], [8], we introduce the time averages

$$\langle q \rangle_T = \frac{1}{T} \int_0^T q \circ \exp(tH_p) dt, \quad T > 0, \quad (1.2)$$

of q along the classical trajectories of p .

The main focus of the works [16] and [15] was on the case when the H_p -flow is completely integrable. The real energy surface $p^{-1}(0)$ is then foliated by invariant Lagrangian tori, along with possibly some other more complicated invariant sets. Given an invariant torus $\Lambda \subset p^{-1}(0)$, which is Diophantine (i.e. the rotation number of the H_p -flow along Λ is poorly approximated by rational numbers), or more generally, irrational, then the time averages $\langle q \rangle_T$ along Λ converge to the space average $\langle q \rangle(\Lambda)$ of q over Λ , as $T \rightarrow \infty$. When Λ is a torus with a rational rotation number, or a more general singular invariant set in the foliation of the energy surface $p^{-1}(0)$, then we need to consider the whole interval $Q_\infty(\Lambda)$ of limits of the flow averages above.

The principal result of [16] says, somewhat roughly, that if $F_0 \in \mathbf{R}$ is a value such that $F_0 = \langle q \rangle(\Lambda_0)$ for a single Diophantine torus $\Lambda_0 \subset p^{-1}(0)$, and F_0 does not belong to $Q_\infty(\Lambda)$ for any other invariant set Λ in the energy surface, then the spectrum of P_ε can be completely determined in a rectangle $[-h^\delta/C, h^\delta/C] + i\varepsilon[F_0 - h^\delta/C, F_0 + h^\delta/C]$ modulo $\mathcal{O}(h^\infty)$, where δ is a positive exponent that can be chosen arbitrarily small, and ε may vary in any interval of the form $h^K < \varepsilon \ll 1$. The spectrum has a structure of a distorted lattice, with horizontal spacing $\sim h$ and vertical spacing $\sim \varepsilon h$. In the work [15], we continued the analysis of the completely integrable case by investigating what happens when the value F_0 belongs in addition to finitely many

intervals $Q_\infty(\Lambda)$, corresponding to rational invariant tori Λ . It was shown in [15] that the number of eigenvalues that can be created by such tori is much smaller than the number of eigenvalues coming from the Diophantine ones, provided that the strength ε of the non-selfadjoint perturbation satisfies $h \ll \varepsilon \leq h^{2/3+\delta}$, $\delta > 0$, and assuming that $F_0 \in Q_\infty(\Lambda) \setminus \langle q \rangle(\Lambda)$, for each rational torus Λ .

The purpose of the present paper is to investigate the global distribution of the imaginary parts of eigenvalues of P_ε in an intermediate spectral band, bounded from above and from below by levels such as F_0 , described above. In fact, when doing so, we shall refrain from treating the more general configurations considered in [15], with both Diophantine and rational tori present, and shall concentrate instead on the simpler situation of [16], where only Diophantine invariant tori corresponding to F_0 , occur. That such a study is planned by the authors was mentioned already in the introduction of [16], and here we are finally able to present the result, giving a Weyl type asymptotic formula for the number of eigenvalues of P_ε in such a spectral band. Roughly speaking, the main result of the present paper is as follows: let $F_j \in \mathbf{R}$, $j = 1, 3$, $F_3 < F_1$, be such that $F_j = \langle q \rangle(\Lambda_j)$, where $\Lambda_j \subset p^{-1}(0)$ are Diophantine tori, and assume that F_j does not belong to $Q_\infty(\Lambda)$ when $\Lambda_j \neq \Lambda$ is an invariant set, $j = 1, 3$. Let $E_2 < 0 < E_4$ be close enough to $0 \in \mathbf{R}$. Then the number of eigenvalues of P_ε in the rectangle $[E_2, E_4] + i\varepsilon[F_3, F_1]$ is equal to

$$\frac{1}{(2\pi h)^2} \text{vol} \left(\bigcup_{E_2 \leq E \leq E_4} \Omega(E) \right) (1 + o(1)), \quad h \rightarrow 0.$$

Here the set

$$\Omega(0) = \{\rho \in p^{-1}(0), Q_\infty(\Lambda(\rho)) \subset [F_3, F_1]\}$$

is flow-invariant, with $\partial\Omega(0) = \Lambda_1 \cup \Lambda_3$. The flow-invariant sets $\Omega(E) \subset p^{-1}(E)$, $E_2 \leq E \leq E_4$ are defined similarly — see the precise statement of Theorem 2.1 below. We may say therefore that the imaginary parts of the eigenvalues of P_ε are distributed according to a Weyl law, expressed in terms of the long-time averages of the leading non-selfadjoint perturbation q along the classical flow of p .

We would like to conclude the introduction by mentioning the work of Shnirelman [27] in the two-dimensional KAM-type situation, which contains the idea of exploiting invariant Lagrangian tori to separate the real energy surface into different invariant regions, in order to study the asymptotic multiplicity of the spectrum of the Laplacian. See also [5]. In our non-selfadjoint case, the idea of using invariant tori as barriers for the flow becomes more efficient than in the standard selfadjoint setting, and the main result of this work can be considered as a justification of this statement.

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2 Assumptions and statement of main result

2.1 General assumptions

We shall describe the general assumptions on our operators, which will be the same as in [16], [15], as well as in the earlier papers in this series. Let M stand for either the space \mathbf{R}^2 or a real analytic compact manifold of dimension two. We shall let \widetilde{M} denote a complexification of M , so that $\widetilde{M} = \mathbf{C}^2$ in the Euclidean case, and in the compact case, we let \widetilde{M} be a Grauert tube of M — see [9] for the definition and further references.

When $M = \mathbf{R}^2$, let

$$P_\varepsilon = P^w(x, hD_x, \varepsilon; h), \quad 0 < h \leq 1, \quad (2.1)$$

be the h -Weyl quantization on \mathbf{R}^2 of a symbol $P(x, \xi, \varepsilon; h)$ (i.e. the Weyl quantization of $P(x, h\xi, \varepsilon; h)$), depending smoothly on $\varepsilon \in \text{neigh}(0, [0, \infty))$ and taking values in the space of holomorphic functions of (x, ξ) in a tubular neighborhood of \mathbf{R}^4 in \mathbf{C}^4 , with

$$|P(x, \xi, \varepsilon; h)| \leq \mathcal{O}(1)m(\text{Re}(x, \xi)), \quad (2.2)$$

there. Here $m \geq 1$ is an order function on \mathbf{R}^4 , in the sense that

$$m(X) \leq C_0 \langle X - Y \rangle^{N_0} m(Y), \quad X, Y \in \mathbf{R}^4, \quad (2.3)$$

for some fixed $C_0, N_0 > 0$. We shall assume, as we may, that m belongs to its own symbol class, so that $m \in C^\infty(\mathbf{R}^4)$ and $\partial^\alpha m = \mathcal{O}_\alpha(m)$ for each $\alpha \in \mathbf{N}^4$.

Assume furthermore that as $h \rightarrow 0$,

$$P(x, \xi, \varepsilon; h) \sim \sum_{j=0}^{\infty} h^j p_{j,\varepsilon}(x, \xi) \quad (2.4)$$

in the space of holomorphic functions satisfying (2.2) in a fixed tubular neighborhood of \mathbf{R}^4 . We make the basic assumption of ellipticity near infinity,

$$|p_{0,\varepsilon}(x, \xi)| \geq \frac{1}{C} m(\text{Re}(x, \xi)), \quad |(x, \xi)| \geq C, \quad (2.5)$$

for some $C > 0$.

When M is a compact manifold, we shall take P_ε to be an h -differential operator on M , such that for every choice of local coordinates, centered at some point of M , it takes the form

$$P_\varepsilon = \sum_{|\alpha| \leq m} a_{\alpha, \varepsilon}(x; h) (hD_x)^\alpha, \quad (2.6)$$

where $a_{\alpha, \varepsilon}(x; h)$ is a smooth function of $\varepsilon \in \text{neigh}(0, [0, \infty))$ with values in the space of bounded holomorphic functions in a complex neighborhood of $x = 0$. We further assume that

$$a_{\alpha, \varepsilon}(x; h) \sim \sum_{j=0}^{\infty} a_{\alpha, \varepsilon, j}(x) h^j, \quad h \rightarrow 0, \quad (2.7)$$

in the space of such functions. The semiclassical principal symbol $p_{0, \varepsilon}$, defined on T^*M , takes the form

$$p_{0, \varepsilon}(x, \xi) = \sum a_{\alpha, \varepsilon, 0}(x) \xi^\alpha, \quad (2.8)$$

if (x, ξ) are canonical coordinates on T^*M . We make the ellipticity assumption,

$$|p_{0, \varepsilon}(x, \xi)| \geq \frac{1}{C} \langle \xi \rangle^m, \quad (x, \xi) \in T^*M, \quad |\xi| \geq C, \quad (2.9)$$

for some large $C > 0$. Here we assume that M has been equipped with some real analytic Riemannian metric, so that $|\xi|$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ are well-defined. We shall consider the operator P_ε as an unbounded operator: $L^2(M, \mu(dx)) \rightarrow L^2(M, \mu(dx))$, with the domain $H^m(M)$, the standard Sobolev space of order m . Here $\mu(dx)$ is the Riemannian volume element on M .

Back in the Euclidean case, the ellipticity assumption (2.5) implies that, for $h > 0$ small enough and when equipped with the domain

$$H(m) := (m^w(x, hD))^{-1} (L^2(\mathbf{R}^2)), \quad (2.10)$$

the operator P_ε becomes closed and densely defined on $L^2(\mathbf{R}^2)$. We shall furthermore make the following assumption concerning the growth of the order function $m(X)$ at infinity: for some $k \in \mathbf{N}$, we have

$$\frac{1}{m^k} \in L^1(\mathbf{R}^4). \quad (2.11)$$

Since P_ε can be replaced by its k -th power P_ε^k in what follows, we may and will assume henceforth that $k = 1$. The assumption that

$$\frac{1}{m} \in L^1(\mathbf{R}^4) \quad (2.12)$$

will prove very useful in various trace class considerations throughout the paper. In the compact case, it amounts to assuming that the order m of P_ε satisfies $m > 2$.

We shall write p_ε for $p_{0,\varepsilon}$ and simply p for $p_{0,0}$. We make the assumption that

$$P_{\varepsilon=0} \quad \text{is formally selfadjoint.} \quad (2.13)$$

It follows from the assumptions (2.5), (2.9), and (2.13) that the spectrum of P_ε in a fixed neighborhood of $0 \in \mathbf{C}$ is discrete, when $0 < h \leq h_0$, $0 \leq \varepsilon \leq \varepsilon_0$, with $h_0 > 0$, $\varepsilon_0 > 0$ sufficiently small. Moreover, if $z \in \text{neigh}(0, \mathbf{C})$ is an eigenvalue of P_ε then $\text{Im } z = \mathcal{O}(\varepsilon)$.

We furthermore assume that the real energy surface $p^{-1}(0) \cap T^*M$ is connected and that

$$dp \neq 0 \quad \text{along} \quad p^{-1}(0) \cap T^*M. \quad (2.14)$$

We then have the following general result, stating that the distribution of the real parts the eigenvalues of P_ε near $0 \in \mathbf{C}$ is governed by the usual Weyl law of the form (1.1), with the following remainder estimate,

$$\#\{z \in \text{Spec}(P_\varepsilon); E_1 \leq \text{Re } z \leq E_2\} = \frac{1}{(2\pi h)^n} \iint_{p^{-1}([E_1, E_2])} dx \, d\xi + \mathcal{O}(\max(\varepsilon, h)h^{-n}). \quad (2.15)$$

Here $E_j \in \text{neigh}(0, \mathbf{R})$, $j = 1, 2$, are close enough to 0 but independent of h . In fact, in (2.15) we may allow E_j to depend on h with $E_2 - E_1 \geq c \max(\varepsilon, h)$, for some $c > 0$ fixed. The Weyl law (2.15) is obtained by following the general arguments of section 5 of [33], which are based on an adaptation to the semiclassical case of the ideas in [22], [23]. See also [1].

Let $H_p = p'_\xi \cdot \partial_x - p'_x \cdot \partial_\xi$ be the Hamilton field of p . Throughout this paper, we shall work under the assumption that the H_p -flow is completely integrable. Following [15], let us now proceed to discuss the precise assumptions on the geometry of the energy surface $p^{-1}(0) \cap T^*M$ in this case.

2.2 Assumptions related to the complete integrability

We assume that there exists an analytic real valued function f defined near $p^{-1}(0) \cap T^*M$ such that $H_p f = 0$, with the differentials df and dp being linearly independent on an open and dense set $\subset \text{neigh}(p^{-1}(0) \cap T^*M, T^*M)$. For each $E \in \text{neigh}(0, \mathbf{R})$, the level sets

$$\Lambda_{\mu, E} = f^{-1}(\mu) \cap p^{-1}(E) \cap T^*M$$

are invariant under the H_p -flow and form a singular foliation of the three-dimensional hypersurface $p^{-1}(E) \cap T^*M$. When $(\mu, E) \in \mathbf{R}^2$ is such that $df \wedge dp \neq 0$ along $\Lambda_{\mu, E}$, then $\Lambda_{\mu, E}$ is a two-dimensional real analytic Lagrangian submanifold of T^*M , which is a finite union of tori. In what follows we shall use the word "leaf" and notation Λ for

a connected component of some $\Lambda_{\mu,E}$. Let J be the set of all leaves in $p^{-1}(0) \cap T^*M$. We have a disjoint union decomposition

$$p^{-1}(0) \cap T^*M = \bigcup_{\Lambda \in J} \Lambda, \quad (2.16)$$

where Λ are compact flow-invariant connected sets.

The set J has a natural structure of a graph whose edges correspond to families of regular leaves and the set S of vertices is composed of singular leaves. The union of edges $J \setminus S$ possesses a natural real analytic structure and the corresponding Lagrangian tori depend analytically on $\Lambda \in J \setminus S$ with respect to that structure.

For simplicity, we shall assume that f is a Morse-Bott function restricted to $p^{-1}(0) \cap T^*M$. In this case, the structure of the singular leaves is known [2], [39], and the set J is a finite connected graph. We shall identify each edge of J analytically with a real bounded interval and this determines a distance on J in the natural way. The following continuity property holds,

$$\begin{aligned} &\text{For every } \Lambda_0 \in J \text{ and every } \varepsilon > 0, \text{ there exists } \delta > 0, \text{ such that if} \quad (2.17) \\ &\Lambda \in J, \text{ dist}_J(\Lambda, \Lambda_0) < \delta, \text{ then } \Lambda \subset \{\rho \in p^{-1}(0) \cap T^*M; \text{dist}(\rho, \Lambda_0) < \varepsilon\}. \end{aligned}$$

By the Arnold-Mineur-Liouville theorem [4], each torus $\Lambda \in J \setminus S$ carries real analytic coordinates x_1, x_2 identifying Λ with $\mathbf{T}^2 := \mathbf{R}^2/2\pi\mathbf{Z}^2$, so that along Λ , we have

$$H_p = a_1 \partial_{x_1} + a_2 \partial_{x_2}, \quad (2.18)$$

where $a_1, a_2 \in \mathbf{R}$. The rotation number is defined as the ratio

$$\omega(\Lambda) = [a_1 : a_2] \in \mathbf{RP}^1, \quad (2.19)$$

and it depends analytically on $\Lambda \in J \setminus S$.

In what follows we shall write

$$p_\varepsilon = p + i\varepsilon q + \mathcal{O}(\varepsilon^2), \quad (2.20)$$

in a neighborhood of $p^{-1}(0) \cap T^*M$, and for simplicity we shall assume throughout this paper that q is real valued on the real domain. (In the general case, we should simply replace q below by $\text{Re } q$.) For each torus $\Lambda \in J \setminus S$, we define the torus average $\langle q \rangle(\Lambda)$ obtained by integrating $q|_\Lambda$ with respect to the natural smooth measure on Λ , and assume that the analytic function $J \setminus S \ni \Lambda \mapsto \langle q \rangle(\Lambda)$ is not identically constant on any open edge. Here the integration measure used in the definition of $\langle q \rangle(\Lambda)$ comes from the diffeomorphism between Λ and \mathbf{T}^2 , given by the action-angle coordinates.

We introduce

$$\langle q \rangle_T = \frac{1}{T} \int_0^T q \circ \exp(tH_p) dt, \quad T > 0, \quad (2.21)$$

and consider the compact intervals $Q_\infty(\Lambda) \subset \mathbf{R}$, $\Lambda \in J$, defined as in [16], [15],

$$Q_\infty(\Lambda) = \left[\lim_{T \rightarrow \infty} \inf_{\Lambda} \langle q \rangle_T, \lim_{T \rightarrow \infty} \sup_{\Lambda} \langle q \rangle_T \right]. \quad (2.22)$$

Notice that when $\Lambda \in J \setminus S$ and $\omega(\Lambda) \notin \mathbf{Q}$ then $Q_\infty(\Lambda) = \{\langle q \rangle(\Lambda)\}$. In the rational case, we write $\omega(\Lambda) = \frac{m}{n}$, where $m \in \mathbf{Z}$ and $n \in \mathbf{N}$ are relatively prime, and where we may assume that $m = \mathcal{O}(n)$. When $k(\omega(\Lambda)) := |m| + |n|$ is the height of $\omega(\Lambda)$, we recall from Proposition 7.1 in [16] that

$$Q_\infty(\Lambda) \subset \langle q \rangle(\Lambda) + \mathcal{O}\left(\frac{1}{k(\omega(\Lambda))^\infty}\right) [-1, 1]. \quad (2.23)$$

Remark. As $J \setminus S \ni \Lambda \rightarrow \Lambda_0 \in S$, the set of all accumulation points of $\langle q \rangle(\Lambda)$ is contained in the interval $Q_\infty(\Lambda_0)$. Indeed, when $\Lambda \in J \setminus S$ and $T > 0$, there exists $\rho = \rho_{T, \Lambda} \in \Lambda$ such that $\langle q \rangle(\Lambda) = \langle q \rangle_T(\rho)$. Therefore, each accumulation point of $\langle q \rangle(\Lambda)$ as $\Lambda \rightarrow \Lambda_0 \in S$, belongs to $[\inf_{\Lambda_0} \langle q \rangle_T, \sup_{\Lambda_0} \langle q \rangle_T]$. The conclusion follows if we let $T \rightarrow \infty$.

2.3 Statement of the main result

From Theorem 7.6 in [16] we recall that

$$\frac{1}{\varepsilon} \text{Im}(\text{Spec}(P_\varepsilon) \cap \{z; |\text{Re } z| \leq \delta\}) \subset \left[\inf_{\Lambda \in J} Q_\infty(\Lambda) - o(1), \sup_{\Lambda \in J} Q_\infty(\Lambda) + o(1) \right], \quad (2.24)$$

as $\varepsilon, h, \delta \rightarrow 0$. Let us also recall from [16] that a torus $\Lambda \in J \setminus S$ is said to be Diophantine if representing $H_p|_\Lambda = a_1 \partial_{x_1} + a_2 \partial_{x_2}$, as in (2.18), we have

$$|a \cdot k| \geq \frac{1}{C_0 |k|^{N_0}}, \quad 0 \neq k \in \mathbf{Z}^2, \quad (2.25)$$

for some fixed $C_0, N_0 > 0$.

Let

$$F_j \in \bigcup_{\Lambda \in J} Q_\infty(\Lambda), \quad j = 1, 3, \quad F_3 < F_1,$$

be such that there exist finitely many Lagrangian tori

$$\Lambda_{1,j}, \dots, \Lambda_{L_j,j} \in J \setminus S, \quad j = 1, 3, \quad (2.26)$$

that are uniformly Diophantine as in (2.25), and such that

$$\langle q \rangle(\Lambda_{k,j}) = F_j \quad \text{for } 1 \leq k \leq L_j, \quad j = 1, 3, \quad (2.27)$$

with

$$d_\Lambda \langle q \rangle (\Lambda_{k,j}) \neq 0, \quad 1 \leq k \leq L_j, \quad j = 1, 3. \quad (2.28)$$

We shall make the following global assumptions, for $j = 1, 3$:

$$F_j \notin \bigcup_{\Lambda \in J \setminus \{\Lambda_{1,j}, \dots, \Lambda_{L_j,j}\}} Q_\infty(\Lambda). \quad (2.29)$$

Here we recall from [15] that the earlier assumptions imply that for $j = 1, 3$, $F_j \notin Q_\infty(\Lambda)$ for $\Lambda_{k,j} \neq \Lambda \in \text{neigh}(\Lambda_{k,j}, J)$, $1 \leq k \leq L_j$.

For notational simplicity only, throughout the following discussion we shall assume that $L_1 = L_3 = 1$ and we shall then write $\Lambda_1 := \Lambda_{1,1}$ and $\Lambda_3 := \Lambda_{1,3}$ for the corresponding Diophantine tori $\subset p^{-1}(0) \cap T^*M$.

Assume that the strength of the non-selfadjoint perturbation ε satisfies

$$h^K \leq \varepsilon \leq h^\delta, \quad \delta > 0, \quad (2.30)$$

for some fixed $K \geq 1$. In the work [16], under the assumptions (2.29), (2.30), for $h > 0$ small enough, we have determined all eigenvalues of $P_\varepsilon \bmod \mathcal{O}(h^\infty)$, in a spectral window of the form

$$\left[-\frac{\varepsilon^{\tilde{\delta}}}{C}, \frac{\varepsilon^{\tilde{\delta}}}{C} \right] + i\varepsilon \left[F_j - \frac{\varepsilon^{\tilde{\delta}}}{C}, F_j + \frac{\varepsilon^{\tilde{\delta}}}{C} \right], \quad j = 1, 3. \quad (2.31)$$

Here $\tilde{\delta} > 0$ is sufficiently small and the constant $C > 0$ is sufficiently large. Recall also that the eigenvalues of P_ε in the region (2.31) form a distorted lattice, and their total number is

$$\sim \frac{\varepsilon^{2\tilde{\delta}}}{h^2}.$$

Our purpose here is to study the distribution of the imaginary parts of the eigenvalues of P_ε in suitable "large" sub-bands of the entire spectral band

$$\{z \in \text{neigh}(0, \mathbf{C}); \text{Im } z = \mathcal{O}(\varepsilon)\}.$$

Specifically, we shall be interested in counting the number of eigenvalues of P_ε in a region of the form

$$\left[-\frac{\varepsilon^{\tilde{\delta}}}{C}, \frac{\varepsilon^{\tilde{\delta}}}{C} \right] + i\varepsilon [F_3, F_1], \quad (2.32)$$

bounded from above and from below by the Diophantine levels F_j , $j = 1, 3$, introduced above.

The following is the main result of this work.

Theorem 2.1 *Let P_ε satisfy the assumptions made in subsections 2.1 and 2.2, and in particular, (2.12). Let $F_j \in \cup_{\Lambda \in J} Q_\infty(\Lambda)$, $j = 1, 3$, $F_3 < F_1$, be such that the assumptions (2.27), (2.28), (2.29) are satisfied. Let $0 < \delta < 1 < K < \infty$ and assume that $h^K \leq \varepsilon \leq h^\delta$. Let $E_2 < 0 < E_4$ satisfy $|E_j| \sim \varepsilon^{\tilde{\delta}}$, where $0 < \tilde{\delta} < 1/3$ is small enough, so that*

$$h \leq \varepsilon^{10\tilde{\delta}} \log \frac{1}{\varepsilon}.$$

Then the number of eigenvalues of P_ε in the rectangle

$$R = (E_2, E_4) + i\varepsilon(F_3, F_1), \quad (2.33)$$

counted with their algebraic multiplicities, is equal to

$$\frac{1}{(2\pi h)^2} \left(\text{vol} \left(\bigcup_{E_2 \leq E \leq E_4} \Omega(E) \right) + \mathcal{O} \left(\varepsilon^{3\tilde{\delta}} \log \frac{1}{\varepsilon} \right) \right). \quad (2.34)$$

Here the set

$$\Omega(0) = \{ \rho \in p^{-1}(0) \cap T^*M; Q_\infty(\Lambda(\rho)) \subset [F_3, F_1] \}, \quad (2.35)$$

*is flow-invariant, whose boundary is $\Lambda_1 \cup \Lambda_3$. We define the set $\Omega(E) \subset p^{-1}(E) \cap T^*M$ to be the set which is close to $\Omega(0)$ with boundary $\Lambda_1(E) \cup \Lambda_3(E)$ where $\Lambda_j(E)$ is the unique invariant torus $\subset p^{-1}(E) \cap T^*M$ close to Λ_j , determined, thanks to (2.28), by the condition $\langle q \rangle(\Lambda_j(E)) = F_j$, $j = 1, 3$.*

Remark. Combining Theorem 2.1 together with the Weyl law (2.15), we see that as $h \rightarrow 0$, the ratio

$$\frac{\#(\text{Spec}(P_\varepsilon) \cap R)}{\#(\text{Spec}(P_\varepsilon) \cap ([E_2, E_4] + i\mathbf{R}))}$$

converges to the expression

$$\frac{\int_{p^{-1}(0)} 1_{\Omega(0)} \mathcal{L}(d(x, \xi))}{\int_{p^{-1}(0)} \mathcal{L}(d(x, \xi))}.$$

Here $\mathcal{L}(d(x, \xi))$ stands for the Liouville measure on $p^{-1}(0)$.

The result of Theorem 2.1 admits a natural extension to the case when the real energy $E \in [E_2, E_4]$ varies in a sufficiently small but h -independent neighborhood of $0 \in \mathbf{R}$, which we now proceed to describe. When doing so, we shall assume that the tori Λ_j , $j = 1, 3$, introduced in (2.26), satisfy the following additional isoenergetic assumption,

$$d_{\Lambda=\Lambda_j} \omega(\Lambda) \neq 0, \quad j = 1, 3. \quad (2.36)$$

Here $\omega(\Lambda)$ is the rotation number of Λ , introduced in (2.19). Let us represent $\Lambda_j \simeq \{\xi = 0\} \subset T^*\mathbf{T}_x^2$ using the action-angle coordinates near Λ_j , so that $p = p(\xi)$, $\omega(\xi) = [\partial_{\xi_1} p(\xi) : \partial_{\xi_2} p(\xi)]$. It follows from (2.36) that the map

$$\text{neigh}(0, \mathbf{R}^2) \ni \xi \mapsto (p(\xi), \omega(\xi)) \in \mathbf{R} \times \mathbf{RP}^1$$

is a local diffeomorphism. There exists therefore an analytic family of Lagrangian tori $\tilde{\Lambda}_j(E) \subset p^{-1}(E) \cap T^*M$, $E \in \text{neigh}(0, \mathbf{R})$, $j = 1, 3$, close to Λ_j , such that $\omega(\tilde{\Lambda}_j(E)) = \omega(\Lambda_j)$, $j = 1, 3$. Let us set when $E \in \text{neigh}(0, \mathbf{R})$,

$$F_j(E) = \langle q \rangle(\tilde{\Lambda}_j(E)), \quad j = 1, 3, \quad (2.37)$$

and notice that an application of Theorem 2.1 together with the results of [16] allow us to conclude that the number of eigenvalues of P_ε in the region

$$\left\{ \text{Re } z \in [E_2, E_4]; F_3(\text{Re } z) \leq \frac{\text{Im } z}{\varepsilon} \leq F_1(\text{Re } z) \right\}, \quad E_2 < 0 < E_4, \quad |E_j| \sim \varepsilon^{\tilde{\delta}}, \quad (2.38)$$

is given by

$$\frac{1}{(2\pi h)^2} \left(\text{vol} \left(\bigcup_{E_2 \leq E \leq E_4} \tilde{\Omega}(E) \right) + \mathcal{O}(\varepsilon^{2\tilde{\delta}}) \right). \quad (2.39)$$

Here $\tilde{\Omega}(0) = \Omega(0)$ in (2.35) and the set $\tilde{\Omega}(E) \subset p^{-1}(E) \cap T^*M$ is close to $\Omega(0)$, with the boundary $\tilde{\Lambda}_1(E) \cup \tilde{\Lambda}_3(E)$. Covering a sufficiently small but fixed open interval $J \subset \mathbf{R}$ containing $0 \in \mathbf{R}$ by $\mathcal{O}(\varepsilon^{-\tilde{\delta}})$ subintervals of length $\mathcal{O}(\varepsilon^{\tilde{\delta}})$ and applying the result of Theorem 2.1 in the form (2.39) uniformly as $E \in J$ varies, we get the following conclusion, by summing the individual contributions from the subintervals.

Theorem 2.2 *Let us keep all the assumptions of Theorem 2.1 and assume in addition that the isoenergetic condition (2.36) holds. Let $C > 0$ be sufficiently large and let $E_2 < 0 < E_4$ be independent of h with $|E_j| \leq 1/C$, $j = 2, 4$. Introduce the functions*

$$F_j(E) = \langle q \rangle(\tilde{\Lambda}_j(E)), \quad E \in \text{neigh}(0, \mathbf{R}),$$

*for $j = 1, 3$, where $\tilde{\Lambda}_j(E) \subset p^{-1}(E) \cap T^*M$ are Lagrangian tori close to Λ_j , such that $\omega(\tilde{\Lambda}_j(E)) = \omega(\Lambda_j)$, $j = 1, 3$. Then the number of eigenvalues of P_ε in the region*

$$\left\{ E_2 \leq \text{Re } z \leq E_4, F_3(\text{Re } z) \leq \frac{\text{Im } z}{\varepsilon} \leq F_1(\text{Re } z) \right\},$$

counted with their algebraic multiplicities, is equal to

$$\frac{1}{(2\pi h)^2} \left(\text{vol} \left(\bigcup_{E_2 \leq E \leq E_4} \tilde{\Omega}(E) \right) + \mathcal{O}(\varepsilon^{\tilde{\delta}}) \right). \quad (2.40)$$

Here $0 < \tilde{\delta} < 1$ satisfies the same smallness condition as in Theorem 2.1.

In the course of the proof of Theorem 2.1, we shall assume, as we may, that the resolvent $(z - P_\varepsilon)^{-1}$ exists when $z \in \gamma := \partial R$. We equip γ with the positive orientation, and decompose

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4, \quad (2.41)$$

where γ_j for $j = 1, 3$, is the part of γ , where $\text{Im } z = \varepsilon F_j$. Correspondingly, we have $\text{Re } z = E_j$ along γ_j , when $j = 2, 4$. We are going to be concerned with the asymptotic behavior of the trace of the spectral projection of P_ε associated to the spectrum of P_ε in R ,

$$\text{tr} \frac{1}{2\pi i} \int_\gamma (z - P_\varepsilon)^{-1} dz, \quad (2.42)$$

giving the number of eigenvalues of P_ε in the rectangle R .

The plan of the paper is as follows. In Section 3, we recall some basic facts and estimates, established in [16], to be used in the proof of the main result. In particular, we recall the main features of the averaging procedure along the H_p -flow, effectively replacing q in (2.20) by $\langle q \rangle_T$ in (2.21), as well as the associated suitable IR-manifold $\subset T^*\bar{M}$, playing the role of the real phase space throughout the proofs. Section 4 is devoted to the analysis of the trace integrals along the vertical segments γ_2, γ_4 , contributing to the trace in (2.42). Following the ideas and methods of [22], [23], [32], [33], [34], here we make use of auxiliary trace class perturbations, constructed so that the perturbed non-selfadjoint operator has gaps in the spectrum. Let us also notice that the analysis of Section 4 is of a general nature and does not depend directly on the paper [16]. It is in Section 5, concerned with the resolvent integrals along the horizontal segments γ_1, γ_3 , that the results of [16] become important. The trace analysis along the horizontal segments proceeds by means of a pseudodifferential partition of unity, and in particular, when understanding the contributions coming from small neighborhoods of the Diophantine tori Λ_j , $j = 1, 3$, we apply the quantum Birkhoff normal form for P_ε , constructed in [16]. Section 5 is concluded by combining the contributions from the different boundary segments to derive a leading term for the trace integral (2.42). The Weyl law (2.34) in Theorem 2.1 then follows when we let the averaging time T become sufficiently large. The brief Section 6 is concerned with the task of deriving an analog of Theorem 2.1 in the case when the Hamilton flow of p is periodic on energy surfaces $p^{-1}(E) \cap T^*M$, $E \in \text{neigh}(0, \mathbf{R})$, which was the main dynamical assumption in the series of works [12]–[14]. A classical Hamiltonian with a periodic flow can be considered as a degenerate case of a completely integrable symbol, and thus, it seemed natural to include this discussion here. In the final Section 7, we apply Theorems 2.1 and 2.2 to a complex perturbation of the semiclassical Laplacian on an analytic simple surface of revolution, and complement the corresponding discussion in [16].

3 Review of some results from [16]

As alluded to in Section 2, the analysis of the trace of the spectral projection in (2.42) will proceed by working in the exponentially weighted h -dependent Hilbert space $H(\Lambda)$, constructed and exploited in [16]. The purpose of this section is therefore to recall briefly the definition and the main features of the weighted space $H(\Lambda)$, following [16], as well as some resolvent estimates for P_ε , when viewed as an unbounded operator on $H(\Lambda)$.

Following the discussion in Section 5 of [16], let us recall the following result.

Proposition 3.1 *Let us keep all the general assumptions of Section 2, and in particular (2.27), (2.28), and (2.29). Let*

$$\kappa_j : \text{neigh}(\Lambda_j, T^*M) \rightarrow \text{neigh}(\xi = 0, T^*\mathbf{T}^2), \quad j = 1, 3, \quad (3.43)$$

be the real and analytic canonical transformation given by the action-angle coordinates near Λ_j , such that $\kappa_j(\Lambda_j) = \{\xi = 0\}$. Then the leading symbol p_ε of P_ε expressed in terms of the coordinates x and ξ on the torus side, takes the form

$$p_j(\xi) + i\varepsilon q_j(x, \xi) + \mathcal{O}(\varepsilon^2), \quad p_j(\xi) = a_j \cdot \xi + \mathcal{O}(\xi^2), \quad j = 1, 3.$$

Define

$$\langle q_j \rangle(\xi) = \frac{1}{(2\pi)^2} \int q_j(x, \xi) dx, \quad \xi \in \text{neigh}(0, \mathbf{R}^2),$$

so that $\langle q_j \rangle(0) = F_j$, and $dp_j(0) = a_j$ and $d\langle q_j \rangle(0)$ are linearly independent, for $j = 1, 3$. Let $0 < \tilde{\varepsilon} \ll 1$ be such that $\tilde{\varepsilon} \gg \max(\varepsilon, h)$. Then there exists a globally defined smooth IR-manifold $\Lambda \subset T^\widetilde{M}$ and smooth Lagrangian tori $\widehat{\Lambda}_1$ and $\widehat{\Lambda}_3 \subset \Lambda$, such that when $\rho \in \Lambda$ is away from an $\tilde{\varepsilon}$ -neighborhood of $\widehat{\Lambda}_1 \cup \widehat{\Lambda}_3$ in Λ then*

$$|\text{Re } P_\varepsilon(\rho)| \geq \frac{\tilde{\varepsilon}}{\mathcal{O}(1)} \quad (3.44)$$

or

$$|\text{Im } P_\varepsilon(\rho) - \varepsilon F_1| \geq \frac{\varepsilon \tilde{\varepsilon}}{\mathcal{O}(1)} \quad \text{and} \quad |\text{Im } P_\varepsilon(\rho) - \varepsilon F_3| \geq \frac{\varepsilon \tilde{\varepsilon}}{\mathcal{O}(1)}. \quad (3.45)$$

*The manifold Λ is $\mathcal{O}(\varepsilon)$ -close to T^*M and agrees with it away from a neighborhood of $p^{-1}(0) \cap T^*M$. We have*

$$P_\varepsilon = \mathcal{O}(1) : H(\Lambda, m) \rightarrow H(\Lambda).$$

When $j = 1, 3$, there exists an elliptic h -Fourier integral operator

$$U_j = \mathcal{O}(1) : H(\Lambda) \rightarrow L_\theta^2(\mathbf{T}^2),$$

such that microlocally near $\widehat{\Lambda}_j$, we have

$$U_j P_\varepsilon = \left(P_j^{(N)}(hD_x, \varepsilon; h) + R_{N+1,j}(x, hD_x, \varepsilon; h) \right) U_j.$$

Here $P_j^{(N)}(hD_x, \varepsilon; h) + R_{N+1,j}(x, hD_x, \varepsilon; h)$ is defined microlocally near $\xi = 0$ in $T^*\mathbf{T}^2$, the full symbol of $P_j^{(N)}(hD_x, \varepsilon; h)$ is independent of x , and

$$R_{N+1,j}(x, \xi, \varepsilon; h) = \mathcal{O}((h, \xi, \varepsilon)^{N+1}).$$

Here the integer N is arbitrarily large but fixed. The leading symbol of $P_j^{(N)}(hD_x, \varepsilon; h)$ is of the form

$$p_j(\xi) + i\varepsilon \langle q_j \rangle(\xi) + \mathcal{O}(\varepsilon^2).$$

Remark. In the work [16], the case of a single Diophantine level F_j has been considered, with the definition of the Hilbert space $H(\Lambda)$ depending on F_j . It is clear however, from the arguments of [16], that the construction of the IR-manifold Λ can be carried out so that it works for both F_1 and F_3 , since the corresponding Diophantine tori Λ_j , $j = 1, 3$, are disjoint. Let us also observe that the tori $\widehat{\Lambda}_j \subset \Lambda$ are $\mathcal{O}(\varepsilon)$ -close to $\Lambda_j \subset p^{-1}(0) \cap T^*M$ in the C^∞ -sense, $j = 1, 3$.

Remark. Let us recall from [16] that the space $L_\theta^2(\mathbf{T}^2)$ of Floquet periodic functions, occurring in the statement of Proposition 2.1, consists of all $u \in L_{\text{loc}}^2(\mathbf{R}^2)$ such that

$$u(x - \nu) = e^{i\theta \cdot \nu} u(x), \quad \theta = \frac{S}{2\pi h} + \frac{k_0}{4}, \quad \nu \in 2\pi \mathbf{Z}^2.$$

Here $S = (S_1, S_2)$ is the pair of the classical actions computed along two suitable fundamental cycles in Λ_j , and the 2-tuple $k_0 \in \mathbf{Z}^2$ stands for the Maslov indices of the corresponding cycles.

Let G_T be an analytic function defined in a neighborhood of $p^{-1}(0) \cap T^*M$, such that

$$H_p G_T = q - \langle q \rangle_T, \tag{3.46}$$

where $T > 0$ is large enough but fixed. As in [16], we solve (3.46) by setting

$$G_T = \int T J_T(-t) q \circ \exp(tH_p) dt, \quad J_T(t) = \frac{1}{T} J\left(\frac{t}{T}\right),$$

where the function J is compactly supported, piecewise linear, with

$$J'(t) = \delta(t) - 1_{[-1,0]}(t).$$

It follows from the results of [16] that there exists a C^∞ canonical transformation

$$\kappa : \text{neigh}(p^{-1}(0), T^*M) \rightarrow \text{neigh}(p^{-1}(0), \Lambda), \tag{3.47}$$

such that, in the case when $M = \mathbf{R}^2$,

$$\kappa(\rho) = \rho + i\varepsilon H_G(\rho) + \mathcal{O}(\varepsilon^2), \quad (3.48)$$

in the C^∞ -sense. Here the function $G \in C_0^\infty(T^*M)$ is such that in a neighborhood of $p^{-1}(0) \cap T^*M$, away from a small but fixed neighborhood of $\Lambda_1 \cup \Lambda_3$, we have $G = G_T$. Near Λ_j , $j = 1, 3$, the function G is of the form $\tilde{G} \circ \kappa_j^{-1}$, where κ_j is the canonical transformation near Λ_j given by the action-angle variables, defined in (3.43), and \tilde{G} is an analytic function defined in a fixed neighborhood of $\xi = 0$, such that

$$H_p \tilde{G} = q - r, \quad r(x, \xi) = \langle q \rangle(\xi) + \mathcal{O}(\xi^N),$$

where $N \in \mathbf{N}$ is arbitrarily large but fixed. We refer to Section 2 of [16] for the details of the construction of the weight function G , also in the compact case.

Remark. From [16], we know that in a complex neighborhood of $p^{-1}(0) \cap T^*M$, away from a small neighborhood of $\Lambda_1 \cup \Lambda_3$, we have

$$\Lambda = \exp(i\varepsilon H_{G_T})(T^*M).$$

We now come to recall the definition of the Hilbert space $H(\Lambda)$. When doing so, let us first concentrate on the case when $M = \mathbf{R}^2$. We shall then take the standard FBI-Bargmann transform

$$Tu(x) = Ch^{-3/2} \int e^{-\frac{1}{2h}(x-y)^2} u(y) dy, \quad C > 0, \quad (3.49)$$

and remark that according to [30], for a suitable choice of $C > 0$ in (3.49), T maps $L^2(\mathbf{R}^2)$ unitarily onto

$$H_{\Phi_0}(\mathbf{C}^2) := \text{Hol}(\mathbf{C}^2) \cap L^2(\mathbf{C}^2; e^{-\frac{2\Phi_0}{h}} L(dx)).$$

Here $\Phi_0(x) = \frac{1}{2}(\text{Im } x)^2$ and $L(dx)$ is the Lebesgue measure in \mathbf{C}^2 .

When viewing T in (3.49) as a Fourier integral operator with a complex quadratic phase, we introduce the associated complex linear canonical transformation

$$\kappa_T : (y, \eta) \mapsto (x, \xi) = (y - i\eta, \eta), \quad (3.50)$$

mapping the real phase space \mathbf{R}^4 onto the linear IR-manifold

$$\Lambda_{\Phi_0} = \left\{ \left(x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x} \right); x \in \mathbf{C}^2 \right\}.$$

It was shown in [16] that the representation

$$\kappa_T(\Lambda) = \left\{ \left(x, \frac{2}{i} \frac{\partial \Phi_\varepsilon}{\partial x} \right); x \in \mathbf{C}^2 \right\} =: \Lambda_{\Phi_\varepsilon} \quad (3.51)$$

holds. Here the function $\Phi_\varepsilon \in C^\infty(\mathbf{C}^2; \mathbf{R})$ is uniformly strictly plurisubharmonic, with $\Phi_\varepsilon - \Phi_0$ compactly supported, and such that uniformly on \mathbf{C}^2 ,

$$\Phi_\varepsilon(x) - \Phi_0(x) = \mathcal{O}(\varepsilon), \quad \nabla(\Phi_\varepsilon(x) - \Phi_0(x)) = \mathcal{O}(\varepsilon).$$

The Hilbert space $H(\Lambda)$ is then defined so that it agrees with $L^2(\mathbf{R}^2)$ as a linear space, and it is equipped with the norm

$$\|u\| = \|u\|_{H(\Lambda)} := \|Tu\|_{L^2_{\Phi_\varepsilon}}, \quad L^2_{\Phi_\varepsilon} = L^2\left(\mathbf{C}^2; e^{-\frac{2\Phi_\varepsilon}{h}} L(dx)\right). \quad (3.52)$$

Furthermore, the natural Sobolev space $H(\Lambda, m) \subset H(\Lambda)$, associated to Λ and the order function m is introduced so $H(\Lambda, m)$ agrees with the space $H(m)$ in (2.10) as a space, and it is equipped with the norm

$$\|u\|_{H(\Lambda, m)}^2 = \int |Tu(x)|^2 \tilde{m}^2(x) e^{-\frac{2\Phi_\varepsilon(x)}{h}} L(dx), \quad (3.53)$$

where $\tilde{m} = m \circ \kappa_T^{-1}$ is viewed as a function on \mathbf{C}_x^2 in the natural way.

Remark. We notice that in view of (2.12), the inclusion map $H(\Lambda, m) \rightarrow H(\Lambda)$ is of trace class.

When defining the action of the operator P_ε on $H(\Lambda)$, we follow [24], [29], and perform a contour deformation in the integral representation of P_ε on the FBI–Bargmann transform side. The operator P_ε then receives a leading symbol given by $p_\varepsilon|_\Lambda$. More generally, when $a \in S(\Lambda_{\Phi_\varepsilon}, \hat{m})$, with an order function \hat{m} on $\Lambda_{\Phi_\varepsilon} \simeq \mathbf{C}_x^2$, we would like to define the Weyl quantization of $\tilde{a} = a \circ \kappa_T \in C^\infty(\Lambda)$, acting on $H(\Lambda)$. To this end, it will be convenient to introduce a globally unitary h -Fourier integral operator with a complex phase,

$$U = U_\varepsilon : L^2(\mathbf{R}^2) \rightarrow H(\Lambda), \quad U_{\varepsilon=0} = 1, \quad (3.54)$$

depending smoothly on ε , associated to the canonical transformation in (3.47), in order to reduce the quantization procedure on Λ to that of Weyl on $T^*\mathbf{R}^2$. When defining the unitary operator U globally, we follow the procedure described in detail in [30], [25]. We can then introduce

$$\text{Op}_h(\tilde{a}) = \mathcal{O}(1) : H(\Lambda, \hat{m}) \rightarrow H(\Lambda), \quad (3.55)$$

defined as

$$\text{Op}_h(\tilde{a}) = U \circ (\tilde{a} \circ \kappa)^w(x, hD_x) \circ U^*, \quad (3.56)$$

using the h -Weyl quantization on \mathbf{R}^2 . In what follows, when quantizing symbols defined on the IR-manifold Λ , we shall always use the unitary operator U to reduce to the standard phase space $T^*\mathbf{R}^2$.

Remark. In the case when M is compact, the definition of the spaces $H(\Lambda)$ and $H(\Lambda, m) = H(\Lambda, \langle \alpha_\xi \rangle^m)$ has been given in the appendix of [12], following [31]. We refer to the latter paper for a discussion of the action of h -differential operators with analytic coefficients on $H(\Lambda)$. When quantizing a symbol $a \in C_0^\infty(\Lambda)$, we follow [31] and use the Toeplitz quantization, as explained in that paper.

The following result is a consequence of Proposition 3.1 and the arguments of Section 5 of [16]. See also Proposition 6.3 of [15].

Proposition 3.2 *Assume that (2.29) holds and that $h^K \leq \varepsilon \leq h^\delta$, $0 < \delta < 1 < K$. Let $0 < \tilde{\delta} \ll 1$ be sufficiently small and let $N_0 \geq 1$ be fixed. Assume that*

$$z \in \left[-\frac{\varepsilon^{\tilde{\delta}}}{C}, \frac{\varepsilon^{\tilde{\delta}}}{C} \right] + i\varepsilon \left[F_j - \frac{\varepsilon^{\tilde{\delta}}}{C}, F_j + \frac{\varepsilon^{\tilde{\delta}}}{C} \right], \quad j = 1, 3,$$

is such that $\text{dist}(z, \text{Spec}(P_\varepsilon)) \geq \varepsilon h^{N_0}$. We then have, in the sense of linear continuous operators: $H(\Lambda) \rightarrow H(\Lambda, m)$,

$$\| (z - P_\varepsilon)^{-1} \| \leq \frac{\mathcal{O}(1)}{\varepsilon h^{N_0}}. \quad (3.57)$$

4 Trace class perturbations and the vertical boundary segments

In this section we shall be concerned with the trace of the integrals

$$\frac{1}{2\pi i} \int_{\gamma_j} (z - P_\varepsilon)^{-1} dz, \quad j = 2, 4,$$

and, to fix the ideas, we shall take $j = 2$. It will be clear that the treatment of the case $j = 4$ is similar. Furthermore, for simplicity, we shall concentrate on the case when $M = \mathbf{R}^2$.

In the Hilbert space $H(\Lambda)$, let us write

$$P_\varepsilon = P + \mathcal{O}(\varepsilon) : H(\Lambda, m) \rightarrow H(\Lambda),$$

where $P = UP_{\varepsilon=0}U^*$ is selfadjoint in $H(\Lambda)$. Here $P_{\varepsilon=0}$ is selfadjoint on $L^2(\mathbf{R}^2)$. We shall write $p \in C^\infty(\Lambda)$ for the leading symbol of P .

Following the approach of [22],[33], we shall introduce a trace class perturbation of P_ε in order to create a gap in the spectrum. To this end, let $\chi \in C_0^\infty(\mathbf{R}; [0, 1])$ be such that $\chi(t) = 1$ for $|t| \leq 1$, and consider the symbol

$$k(\rho; h) = i\chi \left(\frac{p(\rho) - E_2}{\varepsilon^{3\tilde{\delta}}} \right), \quad \rho \in \Lambda.$$

We shall assume that $0 < \tilde{\delta} < 1$ is so small that

$$\varepsilon^{3\tilde{\delta}} \geq h^{\frac{1}{2}-\eta}, \quad (4.1)$$

for some $\eta > 0$. Associated to $k(\rho; h)$, let us introduce

$$\tilde{P} = P + \varepsilon^{3\tilde{\delta}} K, \quad (4.2)$$

where K is the Weyl quantization of $k(\rho; h)$, obtained by using the h -Weyl quantization on \mathbf{R}^2 , as described in (3.56).

The hypothesis (4.1) together with the standard estimates for the operator norm and the trace class norm of a pseudodifferential operator on \mathbf{R}^n , given in [6], show that

$$\|\tilde{P} - P\| \leq \mathcal{O}(\varepsilon^{3\tilde{\delta}}), \quad \|\tilde{P} - P\|_{\text{tr}} \leq \mathcal{O}\left(\frac{\varepsilon^{6\tilde{\delta}}}{h^2}\right). \quad (4.3)$$

Write next

$$\tilde{P}_\varepsilon = P_\varepsilon + \varepsilon^{3\tilde{\delta}} K.$$

We shall make use of the following crude parametrix construction for $z - \tilde{P}_\varepsilon$, when $\text{Re } z = E_2$ and $\text{Im } z = \mathcal{O}(\varepsilon)$. In doing so let us consider

$$e(\rho, z, \varepsilon) = \frac{1}{z - \tilde{p}_\varepsilon(\rho)}, \quad \tilde{p}_\varepsilon(\rho) := p_\varepsilon(\rho) + i\varepsilon^{3\tilde{\delta}} \chi\left(\frac{p(\rho) - E_2}{\varepsilon^{3\tilde{\delta}}}\right). \quad (4.4)$$

Here p_ε is the leading symbol of P_ε , acting on $H(\Lambda)$. We shall first restrict the attention to the region where

$$|p(\rho) - E_2| \leq \mathcal{O}(\varepsilon^{3\tilde{\delta}}). \quad (4.5)$$

Here we have, with $\text{Re } z = E_2$, $\text{Im } z = \mathcal{O}(\varepsilon)$,

$$|\tilde{p}_\varepsilon(\rho) - z| \geq \frac{\varepsilon^{3\tilde{\delta}}}{\mathcal{O}(1)}. \quad (4.6)$$

Considering the usual expression for $\nabla^\ell e$, $\ell \geq 1$, given by the Faà di Bruno's formula [21], we see that we have to estimate the expression

$$\frac{1}{(z - \tilde{p}_\varepsilon)} \prod_{j=1}^k \frac{\nabla^{\ell_j} \tilde{p}_\varepsilon}{(z - \tilde{p}_\varepsilon)}, \quad \ell_j \geq 1, \quad \ell_1 + \dots + \ell_k = \ell. \quad (4.7)$$

Distinguishing the cases $\ell_j = 1$ and $\ell_j \geq 2$, and using that $\nabla \tilde{p}_\varepsilon = \mathcal{O}(1)$ together with (4.6), we obtain that in the region where (4.5) holds, we have

$$\nabla^\ell e(\rho, z, \varepsilon) = \varepsilon^{-3\tilde{\delta}} \mathcal{O}_\ell(\varepsilon^{-3\tilde{\delta}\ell}). \quad (4.8)$$

When $\rho \in \Lambda$ is in a bounded set away from the region where (4.5) is valid, the estimate (4.8) improves to the following,

$$\nabla^\ell e(\rho, \varepsilon, z) = \mathcal{O} \left(\frac{1}{|p_\varepsilon(\rho) - z|^{1+\ell}} \right). \quad (4.9)$$

Finally, when ρ is in a neighborhood of infinity, where Λ agrees with \mathbf{R}^4 , we get

$$\nabla^\ell e(\rho, \varepsilon, z) = \mathcal{O}_\ell \left(\frac{1}{m(\rho)} \right). \quad (4.10)$$

Introducing $E(z)$ to be the Weyl quantization of $e(\rho, z, \varepsilon)$, defined using the unitary map (3.54), as described in Section 3, we conclude, using (4.1), (4.8), (4.9), (4.10), that

$$E(z) = \mathcal{O} \left(\frac{1}{\varepsilon^{3\delta}} \right) : H(\Lambda) \rightarrow H(\Lambda, m), \quad (4.11)$$

while the trace class norm of the operator $E(z)$ on $H(\Lambda)$ satisfies

$$\|E(z)\|_{\text{tr}} \leq \frac{\mathcal{O}(1)}{h^2} \iint_K \frac{\mu(d\rho)}{\max(|p_\varepsilon(\rho) - z|, \varepsilon^{3\delta})} \leq \frac{\mathcal{O}(1)}{h^2} \log \frac{1}{\varepsilon}. \quad (4.12)$$

Here $\mu(d\rho)$ is the symplectic volume element on the IR-manifold Λ , so that

$$\mu(d\rho) = \frac{\sigma^2}{2!} \Big|_\Lambda,$$

where σ is the complex symplectic $(2, 0)$ -form on $T^*\widetilde{M}$, and $K \subset \Lambda$ is a sufficiently large fixed compact set.

It is then clear from (4.1), (4.8), (4.9), and (4.10), that

$$(z - \widetilde{P}_\varepsilon)E(z) = 1 + R, \quad (4.13)$$

where

$$R = \mathcal{O} \left(\frac{h}{\varepsilon^{6\delta}} \right) : H(\Lambda) \rightarrow H(\Lambda). \quad (4.14)$$

It follows therefore that in the region where $\text{Re } z = E_2$, $\text{Im } z = \mathcal{O}(\varepsilon)$, the operator $z - \widetilde{P}_\varepsilon : H(\Lambda, m) \rightarrow H(\Lambda)$ is bijective, and

$$(z - \widetilde{P}_\varepsilon)^{-1} = \mathcal{O} \left(\frac{1}{\varepsilon^{3\delta}} \right) : H(\Lambda) \rightarrow H(\Lambda, m). \quad (4.15)$$

Furthermore,

$$(z - \widetilde{P}_\varepsilon)^{-1} = E(z)(1 + R)^{-1},$$

and writing $(1 + R)^{-1} = 1 - (1 + R)^{-1}R$, we get

$$\left(z - \tilde{P}_\varepsilon\right)^{-1} - E(z) = -E(z)(1 + R)^{-1}R. \quad (4.16)$$

It follows therefore from (4.12) and (4.14) that the trace class norm of the operator in the left hand side of (4.16) is

$$\mathcal{O}\left(\frac{h\varepsilon^{-6\tilde{\delta}}}{h^2} \log \frac{1}{\varepsilon}\right), \quad (4.17)$$

assuming that (4.1) holds.

We conclude that

$$\mathrm{tr} \frac{1}{2\pi i} \int_{\gamma_2} \left(z - \tilde{P}_\varepsilon\right)^{-1} dz = \mathrm{tr} \frac{1}{2\pi i} \int_{\gamma_2} E(z) dz + \mathcal{O}\left(\frac{h\varepsilon^{-5\tilde{\delta}}}{h^2} \log \frac{1}{\varepsilon}\right). \quad (4.18)$$

Here we have also used that the length of γ_2 is $\mathcal{O}(\varepsilon)$.

We shall now compare traces of the integrals of the resolvents of P_ε and \tilde{P}_ε , following some classical methods in non-selfadjoint spectral theory. Let us also remark that such methods now also have a tradition in the theory of resonances. See [7], [22], [23], [32], [34], [26], [38], [42]. We shall therefore only recall the main features of the argument, referring to the above mentioned works for the details.

Assume that $\mathrm{Re} z = E_2$ and $\mathrm{Im} z = \mathcal{O}(\varepsilon)$. An application of the resolvent identity shows that

$$(z - P_\varepsilon)^{-1} - (z - \tilde{P}_\varepsilon)^{-1} = -(z - P_\varepsilon)^{-1} \varepsilon^{3\tilde{\delta}} K \left(z - \tilde{P}_\varepsilon\right)^{-1}$$

is of trace class on $H(\Lambda)$, and using the cyclicity of the trace, we obtain, by a classical calculation, that

$$\mathrm{tr} \left((z - P_\varepsilon)^{-1} - (z - \tilde{P}_\varepsilon)^{-1} \right) = \mathrm{tr} \left(\left(1 + \tilde{K}(z)\right)^{-1} \partial_z \tilde{K}(z) \right).$$

Here $\tilde{K}(z) = \varepsilon^{3\tilde{\delta}} K \left(z - \tilde{P}_\varepsilon\right)^{-1}$. Hence,

$$\begin{aligned} \mathrm{tr} \left(\frac{1}{2\pi i} \int_{\gamma_2} (z - P_\varepsilon)^{-1} dz - \frac{1}{2\pi i} \int_{\gamma_2} (z - \tilde{P}_\varepsilon)^{-1} dz \right) \\ = \frac{1}{2\pi i} \int_{\gamma_2} \partial_z \log \det \left(1 + \varepsilon^{3\tilde{\delta}} K(z - \tilde{P}_\varepsilon)^{-1} \right) dz. \end{aligned}$$

We shall be interested in the real part of the expression above, which is equal to

$$\frac{1}{2\pi} \text{var arg}_{\gamma_2} D(z),$$

where we have set

$$D(z) = \det \left(1 + \tilde{K}(z) \right). \quad (4.19)$$

An application of (4.3) and (4.15) shows that

$$\| \tilde{K}(z) \|_{\text{tr}} \leq \mathcal{O} \left(\frac{\varepsilon^{3\tilde{\delta}}}{h^2} \right), \quad (4.20)$$

in the region where $\text{Re } z = E_2$, $\text{Im } z = \mathcal{O}(\varepsilon)$.

When estimating the argument variation of $D(z)$ along γ_2 , we shall proceed by following the now well established and essentially classical complex analytic argument, described in detail in [22], [32], [34]. (See also [37].) In order to recall its main features, let us consider the holomorphic determinant $D(z)$ in (4.19) in a region of the form

$$R_d = [E_2 - d\varepsilon^{3\tilde{\delta}}, E_2 + d\varepsilon^{3\tilde{\delta}}] + i[-d\varepsilon^{3\tilde{\delta}}, d\varepsilon^{3\tilde{\delta}}], \quad d > 0, \quad (4.21)$$

and notice that the bound (4.20) remains valid for $z \in R_d$. It follows that

$$|D(z)| \leq \exp(\| \tilde{K}(z) \|_{\text{tr}}) \leq \exp \left(\mathcal{O}(1) \frac{\varepsilon^{3\tilde{\delta}}}{h^2} \right), \quad z \in R_d. \quad (4.22)$$

Let now $z_0 \in R_d$ be such that $\text{Im } z_0 < 0$ and $|\text{Im } z_0| \geq d_1 \varepsilon^{3\tilde{\delta}}$, $d_1 < d$. We then have

$$(z_0 - P_\varepsilon)^{-1} = \mathcal{O} \left(\frac{1}{\varepsilon^{3\tilde{\delta}}} \right) : H(\Lambda) \rightarrow H(\Lambda),$$

and therefore, since

$$D(z_0)^{-1} = \det \left((1 + \tilde{K}(z_0))^{-1} \right),$$

with

$$(1 + \tilde{K}(z_0))^{-1} = 1 - \varepsilon^{3\tilde{\delta}} K(z_0 - P_\varepsilon)^{-1},$$

it follows that

$$|D(z_0)| \geq \exp \left(-\mathcal{O}(1) \frac{\varepsilon^{3\tilde{\delta}}}{h^2} \right). \quad (4.23)$$

Let $N = N(P_\varepsilon, R_d, h)$ be the number of eigenvalues z_j , $j = 1, \dots, N$, of P_ε in R_d , repeated according to their multiplicity. Using that (4.20) continues to be valid in a region of the form R_d , with a slightly larger value of d , and combining this with (4.23) and Jensen's formula, we obtain that

$$N(P_\varepsilon, R_d, h) = \mathcal{O}(1) \frac{\varepsilon^{3\tilde{\delta}}}{h^2}. \quad (4.24)$$

Proceeding further as in [22], [32], [34], one next considers a factorization

$$D(z) = G(z) \prod_{j=1}^N (z - z_j), \quad z \in R_d, \quad (4.25)$$

where G and $1/G$ are holomorphic in R_d . An application of Cartan's lemma (or, alternatively, of Lemma 4.3 in [34]) together with the maximum principle and the Harnack inequality allows us to show that after an arbitrarily small decrease of $d > 0$, we have

$$|\log |G(z)|| \leq \mathcal{O} \left(\frac{\varepsilon^{3\tilde{\delta}}}{h^2} \right), \quad z \in R_d.$$

It follows then easily (see, for instance, Lemma 1.8 in [22]) that the argument variation of $G(z)$ along γ_2 is

$$\text{var arg}_{\gamma_2} G(z) = \mathcal{O} \left(\frac{\varepsilon^{3\tilde{\delta}}}{h^2} \right).$$

Combining this with (4.24) and (4.25), we obtain that

$$\text{var arg}_{\gamma_2} D(z) = \mathcal{O} \left(\frac{\varepsilon^{3\tilde{\delta}}}{h^2} \right).$$

Proposition 4.1 *Assume that $\tilde{\delta} > 0$ is small enough so that $h^{\frac{1}{2}-\eta} \leq \varepsilon^{3\tilde{\delta}}$, for some $\eta > 0$. We have*

$$\text{Re} \left(\text{tr} \left(\frac{1}{2\pi i} \int_{\gamma_2} (z - P_\varepsilon)^{-1} dz - \frac{1}{2\pi i} \int_{\gamma_2} (z - \tilde{P}_\varepsilon)^{-1} dz \right) \right) = \mathcal{O} \left(\frac{\varepsilon^{3\tilde{\delta}}}{h^2} \right). \quad (4.26)$$

Combining (4.18) and Proposition 4.1, we obtain that

$$\text{Re tr} \frac{1}{2\pi i} \int_{\gamma_2} (z - P_\varepsilon)^{-1} dz = \text{Re tr} \frac{1}{2\pi i} \int_{\gamma_2} E(z) dz + \mathcal{O} \left(\frac{\varepsilon^{3\tilde{\delta}}}{h^2} + \frac{h\varepsilon^{-5\tilde{\delta}}}{h^2} \log \frac{1}{\varepsilon} \right), \quad (4.27)$$

and here the remainder in the right hand side is

$$\mathcal{O} \left(\frac{\varepsilon^{3\tilde{\delta}}}{h^2} \right),$$

provided that $\tilde{\delta} > 0$ is so small that

$$h \leq \frac{\varepsilon^{8\tilde{\delta}}}{|\log \varepsilon|}. \quad (4.28)$$

Notice that the smallness condition (4.1) is implied by (4.28), provided that $\eta > 0$ in (4.1) is sufficiently small.

Since the operator $E(z)$ is introduced by means of the Weyl quantization on \mathbf{R}^2 and the unitary map (3.54), associated to the canonical transformation (3.47), we know that the trace of the trace class operator $E(z) : H(\Lambda) \rightarrow H(\Lambda)$ is given by

$$\mathrm{tr} E(z) = \frac{1}{(2\pi h)^2} \iint \frac{1}{z - \tilde{p}_\varepsilon(\rho)} \mu(d\rho).$$

Here we recall that \tilde{p}_ε has been introduced in (4.4).

With (4.27) in mind, we shall now compare the expressions

$$\frac{1}{2\pi i} \int_{\gamma_2} \iint \frac{1}{z - p_\varepsilon(\rho)} \mu(d\rho) dz \quad (4.29)$$

and

$$\frac{1}{2\pi i} \int_{\gamma_2} \iint \frac{1}{z - \tilde{p}_\varepsilon(\rho)} \mu(d\rho) dz. \quad (4.30)$$

The difference between the integrals (4.29) and (4.30) is equal to

$$\frac{-1}{2\pi i} \int_{\gamma_2} \iint \frac{i\varepsilon^{3\tilde{\delta}} \chi\left((p - E_2)/\varepsilon^{3\tilde{\delta}}\right)}{(z - \tilde{p}_\varepsilon)(z - p_\varepsilon)} \mu(d\rho) dz, \quad (4.31)$$

which, in view of (4.6), does not exceed

$$\mathcal{O}(1) \int_{\gamma_2} \iint \frac{\chi((p - E_2)/\varepsilon^{3\tilde{\delta}})}{|z - p_\varepsilon|} \mu(d\rho) |dz|,$$

which can in turn be estimated by

$$\mathcal{O}(1) \iint \chi\left(\frac{p - E_2}{\varepsilon^{3\tilde{\delta}}}\right) (-\log |p - E_2|) \mu(d\rho) = \mathcal{O}(1) \int_0^{\varepsilon^{3\tilde{\delta}}} -\log t dt = \mathcal{O}(1) \varepsilon^{3\tilde{\delta}} \log \frac{1}{\varepsilon}.$$

We summarize the discussion in this section in the following proposition.

Proposition 4.2 *Let $E_2 < 0 < E_4$ be such that $|E_j| \sim \varepsilon^{\tilde{\delta}}$, $j = 2, 4$, where $0 < \tilde{\delta} < 1$ is so small that*

$$h \leq \frac{\varepsilon^{8\tilde{\delta}}}{|\log \varepsilon|}.$$

When γ_j is the vertical segment given by $\mathrm{Re} z = E_j$, $\varepsilon F_3 \leq \mathrm{Im} z \leq \varepsilon F_1$, we have, for $j = 2, 4$,

$$\begin{aligned} \mathrm{Re} \mathrm{tr} \frac{1}{2\pi i} \int_{\gamma_j} (z - P_\varepsilon)^{-1} dz \\ = \mathrm{Re} \frac{1}{2\pi i} \frac{1}{(2\pi h)^2} \int_{\gamma_j} \iint \frac{1}{z - p_\varepsilon(\rho)} \mu(d\rho) dz + \frac{1}{h^2} \mathcal{O}\left(\varepsilon^{3\tilde{\delta}} \log \frac{1}{\varepsilon}\right). \end{aligned}$$

Remark. As mentioned above, the idea of using trace class perturbations to create a gap in the spectrum of a non-selfadjoint operator has a long tradition in non-selfadjoint spectral theory, [23], [33]. Here we have chosen to create gaps that are wide enough, so that simple pseudodifferential perturbations can be employed to that end. The price to pay for this simplicity is that the remainder estimates that one obtains in the trace analysis in Proposition 4.2 are not expected to be sharp, and it is quite likely that finer estimates are possible to derive, at the expense of a greater technical investment. We hope to be able to return to this question in a future paper.

5 Trace analysis near the Diophantine levels

The purpose of this section is to understand the semiclassical behavior of the trace integrals

$$\mathrm{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} dz, \quad j = 1, 3,$$

and when doing so, we shall concentrate on the case when $j = 1$. Here we recall that γ_1 is given by $\mathrm{Im} z = \varepsilon F_1$, $E_2 \leq \mathrm{Re} z \leq E_4$, $|E_j| \sim \varepsilon^{\tilde{\delta}}$, $j = 2, 4$. We shall assume throughout that $0 < \tilde{\delta} < 1$ satisfies

$$h \leq \mathcal{O}(\varepsilon^{9\tilde{\delta}}), \tag{5.1}$$

so that Proposition 4.2 is applicable. Recall also that $F_3 < F_1$. We may, and will assume, in the remainder of the following discussion that $z \in \gamma_1$ satisfies

$$\mathrm{dist}(z, \mathrm{Spec}(P_\varepsilon)) \geq \varepsilon h^{N_0}, \tag{5.2}$$

for some fixed $N_0 \geq 1$, so that by Proposition 3.2, the resolvent $(z - P_\varepsilon)^{-1}$ satisfies an estimate of the form (3.57).

5.1 Trace integrals away from the Diophantine tori

Following [16], thanks to Proposition 3.1, we shall consider a smooth partition of unity on the manifold Λ , given by

$$1 = \chi_1 + \chi_3 + \psi_{r,+} + \psi_{r,-} + \psi_{i,-} + \psi_{i,0} + \psi_{i,+}. \tag{5.3}$$

Here $0 \leq \chi_j \in C_0^\infty(\Lambda)$ is a cut-off function to an $\varepsilon^{\tilde{\delta}}$ -neighborhood of $\widehat{\Lambda}_j$, $j = 1, 3$, such that, in the sense of trace class operators on $H(\Lambda)$, we have

$$[P_\varepsilon, \chi_j] = \mathcal{O}(h^M). \tag{5.4}$$

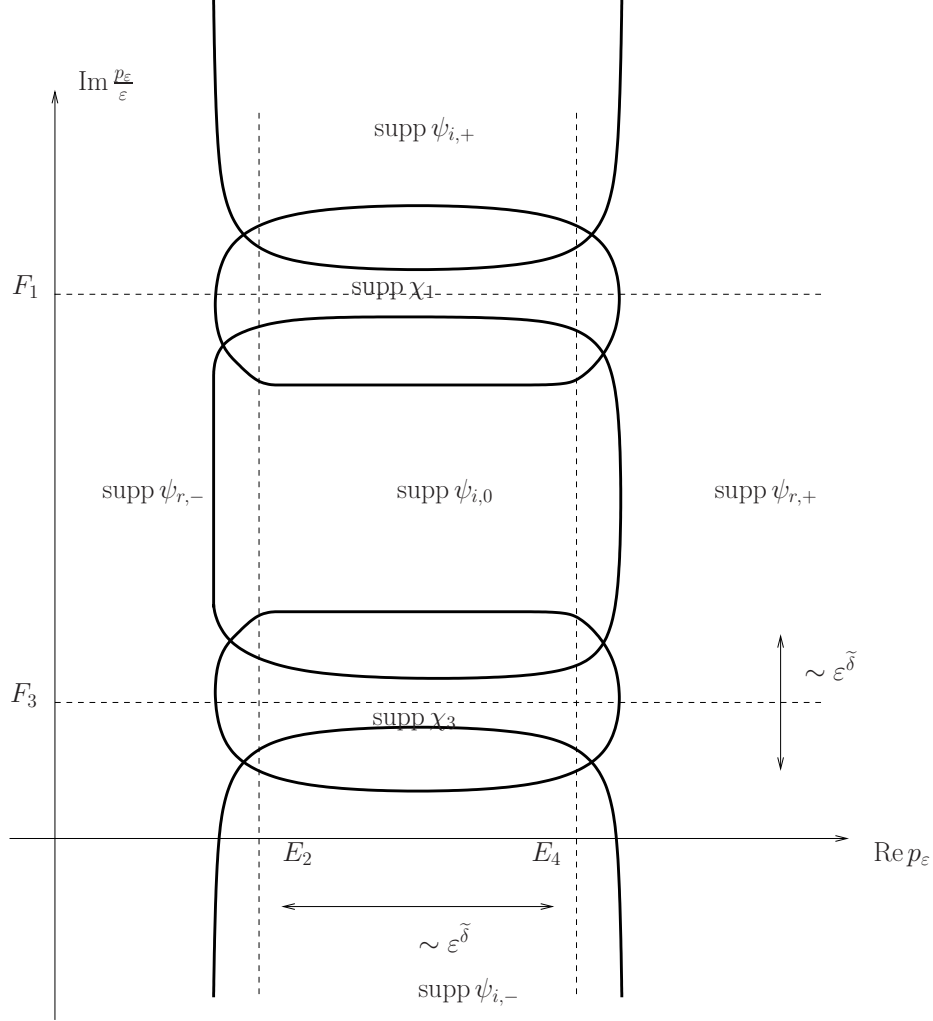


Figure 1: A schematic representation of the partition of unity (5.3) on the IR-manifold Λ , chosen according to Proposition 3.1. Here p_ε is the leading symbol of P_ε , acting on $H(\Lambda)$.

The integer $M = M(N, \delta, \tilde{\delta}) > 0$ is fixed and can be taken arbitrarily large by choosing the integer N in Proposition 3.1 large enough. As observed and exploited in [16], such a choice of the cut-off χ_j is possible thanks to Proposition 3.1.

The functions $0 \leq \psi_{r,\pm}$ in (5.3) are chosen so that

$$\pm \operatorname{Re} P_\varepsilon(\rho) \geq \frac{\varepsilon \tilde{\delta}}{\mathcal{O}(1)}, \quad (5.5)$$

near the support of $\psi_{r,\pm}$, respectively. Here we may assume that the support of $\psi_{r,+}$ contains a neighborhood of infinity, where Λ agrees with T^*M , and there the estimate (5.5) improves to the following one,

$$\operatorname{Re} P_\varepsilon(\rho) \geq \frac{m(\rho)}{\mathcal{O}(1)}. \quad (5.6)$$

We shall now describe the properties of the cut-off functions $\psi_{i,\pm}$ and $\psi_{i,0}$, occurring in (5.3). The function $0 \leq \psi_{i,-} \in C_0^\infty(\Lambda)$ is such that near the support of $\psi_{i,-}$ we have

$$\operatorname{Im} P_\varepsilon \leq \varepsilon F_3 - \frac{\varepsilon \varepsilon \tilde{\delta}}{\mathcal{O}(1)},$$

while near $\operatorname{supp}(\psi_{i,+})$ we have

$$\operatorname{Im} P_\varepsilon \geq \varepsilon F_1 + \frac{\varepsilon \varepsilon \tilde{\delta}}{\mathcal{O}(1)}.$$

Finally, we have

$$\varepsilon F_3 + \frac{\varepsilon \varepsilon \tilde{\delta}}{\mathcal{O}(1)} \leq \operatorname{Im} P_\varepsilon \leq \varepsilon F_1 - \frac{\varepsilon \varepsilon \tilde{\delta}}{\mathcal{O}(1)},$$

near the support of $\psi_{i,0} \in C_0^\infty(\Lambda)$. Continuing to follow [16], we may and will arrange so that in the sense of trace class operators on $H(\Lambda)$, we have

$$A[P_\varepsilon, \psi_{i,\cdot}] = \mathcal{O}(h^M), \quad \cdot = \pm, 0, \quad (5.7)$$

and also,

$$[P_\varepsilon, \psi_{i,\cdot}]A = \mathcal{O}(h^M), \quad \cdot = \pm, 0. \quad (5.8)$$

Here A is a microlocal cut-off to a region where $|\operatorname{Re} P_\varepsilon| \leq \varepsilon \tilde{\delta} / \mathcal{O}(1)$, and $M = M(N, \delta, \tilde{\delta})$ is an integer having the same properties as the integer in (5.4).

Decomposing the operator $(z - P_\varepsilon)^{-1}$ according to (5.3), we shall first analyze the trace of the integral

$$\frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \psi_{r,+} dz.$$

We have

$$(z - P_\varepsilon)^{-1} = \mathcal{O}\left(\frac{1}{\varepsilon h^{N_0}}\right) : H(\Lambda) \rightarrow H(\Lambda, m), \quad (5.9)$$

and from [16], let us recall that

$$(z - P_\varepsilon)^{-1} \psi_{r,+} = \mathcal{O}\left(\frac{1}{\varepsilon^\delta}\right) : H(\Lambda) \rightarrow H(\Lambda, m),$$

provided that $h \leq \mathcal{O}(\varepsilon^{9\tilde{\delta}})$. Thanks to the essentially elliptic estimate (5.5), it is possible to construct a trace class parametrix for $z - P_\varepsilon$, valid near the support of $\psi_{r,+}$. We shall now describe briefly the main steps of this well known construction. Let $\chi = \chi_{r,+} \in C_b^\infty(\Lambda)$ be such that $\operatorname{Re} P_\varepsilon \geq \varepsilon^\delta / \mathcal{O}(1)$ near the support of χ and $\chi = 1$ near $\operatorname{supp}(\psi_{r,+})$. Let

$$e_0(\rho, z, \varepsilon) = \frac{\chi(\rho)}{z - P_\varepsilon(\rho)}.$$

Restricting the attention to a suitable bounded region of Λ , we see that

$$\nabla^\ell e_0 = \mathcal{O}\left(\varepsilon^{-\tilde{\delta}} \varepsilon^{-\tilde{\delta}\ell}\right), \quad \ell \in \mathbf{N},$$

while in a neighborhood of infinity, where Λ agrees with \mathbf{R}^4 , we have

$$\nabla^\ell e_0 = \mathcal{O}\left(\frac{1}{m}\right).$$

It follows that, on the level of operators, we have

$$(z - P_\varepsilon)e_0 = \chi + r_1, \quad r_1 = \mathcal{O}\left(\frac{h}{\varepsilon^{2\tilde{\delta}}}\right) : H(\Lambda) \rightarrow H(\Lambda).$$

Here and in what follows, we are quantizing the symbols on Λ using the Weyl quantization on \mathbf{R}^4 , as explained in Section 3. Notice also that the trace class norm of e_0 does not exceed

$$\frac{1}{h^2} \mathcal{O}\left(\log \frac{1}{\varepsilon}\right).$$

Furthermore, with

$$e_1 = \frac{-r_1 \chi}{z - P_\varepsilon},$$

we get on the level of operators,

$$(z - P_\varepsilon)(e_0 + e_1) = \chi + r_1(1 - \chi) + r_2, \quad r_2 = \mathcal{O}\left(\frac{h^2}{\varepsilon^{4\tilde{\delta}}}\right) : H(\Lambda) \rightarrow H(\Lambda).$$

and continuing in this way we obtain the symbols e_j , $1 \leq j \leq L$, and r_k , $1 \leq k \leq L+1$, $L \in \mathbf{N}$, such that on the level of operators,

$$(z - P_\varepsilon)(e_0 + e_1 + \dots + e_L) = \chi + \sum_{k=1}^L r_k(1 - \chi) + r_{L+1}.$$

Here

$$r_k = \mathcal{O}(h^{M_k}) : H(\Lambda) \rightarrow H(\Lambda),$$

where $M_k \rightarrow \infty$ as $k \rightarrow \infty$. It follows, using also (5.9), that modulo an expression whose trace class norm on $H(\Lambda)$ can be estimated by an arbitrarily high power of h , provided that we take L large enough, we have

$$(z - P_\varepsilon)^{-1} \psi_{r,+} \equiv (e_0 + e_1 + \dots + e_L) \psi_{r,+}.$$

Estimating the trace class norm of $e_j \psi_{r,+}$, $j \geq 1$ and using that the length of γ_1 is $\mathcal{O}(\varepsilon^{\tilde{\delta}})$, we obtain the following result.

Proposition 5.1 *We have*

$$\begin{aligned} & \operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \psi_{r,\pm} dz \\ &= \frac{1}{2\pi i} \frac{1}{(2\pi h)^2} \int_{\gamma_1} \iint \frac{1}{z - p_\varepsilon(\rho)} \psi_{r,\pm}(\rho) \mu(d\rho) dz + \mathcal{O} \left(\varepsilon^{\tilde{\delta}} \frac{h \varepsilon^{-2\tilde{\delta}}}{h^2} \log \frac{1}{\varepsilon} \right). \end{aligned} \quad (5.10)$$

We shall next consider the trace integral

$$\operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \psi_{i,-} dz. \quad (5.11)$$

Here we are no longer in the elliptic region, and in order to understand (5.11), we shall proceed by means of a suitable trace class perturbation, concentrated in the non-elliptic region, followed by a contour deformation argument. To be precise, let \tilde{P}_ε be such that $\tilde{P}_\varepsilon = P_\varepsilon$ near $\operatorname{supp}(\psi_{i,-})$ and with $\operatorname{Im} \tilde{P}_\varepsilon \leq \varepsilon F_3 - \varepsilon \varepsilon^{\tilde{\delta}} / \mathcal{O}(1)$ in the entire region where $|\operatorname{Re} P_\varepsilon| < \varepsilon^{\tilde{\delta}} / \mathcal{O}(1)$, while \tilde{P}_ε agrees with P_ε outside of a slightly larger set of the form $|\operatorname{Re} P_\varepsilon| < \varepsilon^{\tilde{\delta}} / \mathcal{O}(1)$. We shall arrange, as we can, so that the operator \tilde{P}_ε is of the form

$$\tilde{P}_\varepsilon = P_\varepsilon - i\varepsilon\chi, \quad (5.12)$$

where $\chi = \chi(\rho; h) \in C_0^\infty(\Lambda)$ satisfies

$$\nabla^\ell \chi = \mathcal{O}_\ell(\varepsilon^{-\ell\tilde{\delta}}), \quad \ell \geq 0. \quad (5.13)$$

We shall make use of the following result.

Lemma 5.2 Assume that $\tilde{\delta} > 0$ is so small that $h \leq \mathcal{O}(\varepsilon^{9\tilde{\delta}})$, and recall that along γ_1 , the bound $\text{dist}(z, \text{Spec}(P_\varepsilon)) \geq \varepsilon h^{N_0}$ holds, for some fixed $N_0 \geq 1$. We have

1. In the region where $\text{Re } z \in [E_2, E_4]$, $\varepsilon F_1 \leq \text{Im } z \leq \mathcal{O}(\varepsilon^{\tilde{\delta}})$, the operator

$$z - \tilde{P}_\varepsilon : H(\Lambda, m) \rightarrow H(\Lambda)$$

is bijective, with

$$(z - \tilde{P}_\varepsilon)^{-1} = \frac{\mathcal{O}(1)}{\text{Im } z - \varepsilon F_3 + \varepsilon \varepsilon^{\tilde{\delta}}} : H(\Lambda) \rightarrow H(\Lambda, m). \quad (5.14)$$

2. It holds, in the sense of trace class operators on $H(\Lambda)$,

$$(z - \tilde{P}_\varepsilon)^{-1} \psi_{i,-} - (z - P_\varepsilon)^{-1} \psi_{i,-} = \mathcal{O}(h^{\tilde{M}}), \quad z \in \gamma_1, \quad (5.15)$$

where \tilde{M} can be made arbitrarily large, by taking the integer M in (5.7) large enough.

Proof: In the region where $|\text{Re } P_\varepsilon| < \varepsilon^{\tilde{\delta}}/\mathcal{O}(1)$, we have $\text{Im } \tilde{P}_\varepsilon = \mathcal{O}(\varepsilon)$ and

$$\frac{1}{\varepsilon} \text{Im } \tilde{P}_\varepsilon \leq F_3 - \frac{\varepsilon^{\tilde{\delta}}}{\mathcal{O}(1)}.$$

The statement (1) is obtained therefore by a standard application of the sharp Gårding inequality. See also Section 5 of [16] for the details of a similar argument.

When establishing (2), let us notice first that the expression in the left hand side of (5.15) is equal to

$$(z - \tilde{P}_\varepsilon)^{-1} (\tilde{P}_\varepsilon - P_\varepsilon) (z - P_\varepsilon)^{-1} \psi_{i,-}. \quad (5.16)$$

Let $\hat{\psi}_{i,-}$ be such that $\hat{\psi}_{i,-} = 1$ near $\text{supp}(\psi_{i,-})$ and with $\tilde{P}_\varepsilon = P_\varepsilon$ near $\text{supp}(\hat{\psi}_{i,-})$. We can also arrange that (5.7) and (5.8) are also valid for $\hat{\psi}_{i,-}$. Modulo $\mathcal{O}(h^{\tilde{M}})$ in the trace class norm, we may replace the expression in (5.16) by

$$(z - \tilde{P}_\varepsilon)^{-1} (\tilde{P}_\varepsilon - P_\varepsilon) (1 - \hat{\psi}_{i,-}) (z - P_\varepsilon)^{-1} \psi_{i,-},$$

whose trace class norm does not exceed $\mathcal{O}(\varepsilon^{-\tilde{\delta}})$ times the trace class norm of

$$(1 - \hat{\psi}_{i,-}) (z - P_\varepsilon)^{-1} \psi_{i,-}.$$

When estimating the latter, we may follow some arguments of [34]. When $L \in \mathbf{N}$, let

$$\psi_{i,-} := \psi \prec \psi_1 \prec \dots \prec \psi_L \prec \hat{\psi} := \hat{\psi}_{i,-}.$$

Here we arrange also that (5.7) and (5.8) are valid for ψ_j , $1 \leq j \leq L$. Modulo terms that are $\mathcal{O}(h^\infty)$ in the trace class norm, we may write

$$\begin{aligned} (z - P_\varepsilon)^{-1} \psi &\equiv \sum_{j=1}^L \psi_j (z - P_\varepsilon)^{-1} [P_\varepsilon, \psi_{j-1}] \dots (z - P_\varepsilon)^{-1} [P_\varepsilon, \psi_1] (z - P_\varepsilon)^{-1} \psi \\ &\quad + (z - P_\varepsilon)^{-1} [P_\varepsilon, \psi_L] (z - P_\varepsilon)^{-1} [P_\varepsilon, \psi_{L-1}] \dots [P_\varepsilon, \psi_1] (z - P_\varepsilon)^{-1} \psi, \end{aligned}$$

and therefore, still modulo terms that are $\mathcal{O}(h^\infty)$ in the trace class norm, we obtain that

$$\begin{aligned} (1 - \widehat{\psi}) (z - P_\varepsilon)^{-1} \psi &\equiv (1 - \widehat{\psi}) (z - P_\varepsilon)^{-1} [P_\varepsilon, \psi_L] (z - P_\varepsilon)^{-1} [P_\varepsilon, \psi_{L-1}] \dots [P_\varepsilon, \psi_1] (z - P_\varepsilon)^{-1} \psi. \end{aligned}$$

Here we recall that for $1 \leq j \leq L$,

$$A[P_\varepsilon, \psi_j] = \mathcal{O}(h^{\widetilde{M}}), \quad [P_\varepsilon, \psi_j]A = \mathcal{O}(h^{\widetilde{M}}),$$

in the trace class norm, where A is a microlocal cut-off to the region where $|\operatorname{Re} P_\varepsilon| \leq \varepsilon^{\widetilde{\delta}}/\mathcal{O}(1)$. Using also that

$$(z - P_\varepsilon)^{-1} (1 - A) = \mathcal{O}\left(\frac{1}{\varepsilon^{\widetilde{\delta}}}\right) : H(\Lambda) \rightarrow H(\Lambda, m),$$

while the trace class norm of this operator is

$$\frac{1}{h^2} \mathcal{O}\left(\frac{1}{\varepsilon^{\widetilde{\delta}}}\right),$$

as well as the fact that $[P_\varepsilon, \psi_j] = \mathcal{O}(h/\varepsilon^{2\widetilde{\delta}})$, we obtain the second result. \square

An application of Lemma 5.2 shows that

$$\operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \psi_{i,-} dz = \operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - \widetilde{P}_\varepsilon)^{-1} \psi_{i,-} dz + \mathcal{O}(h^{\widetilde{M}}). \quad (5.17)$$

Here

$$\int_{\gamma_1} (z - \widetilde{P}_\varepsilon)^{-1} \psi_{i,-} dz = \int_{\widetilde{\gamma}_1} (z - \widetilde{P}_\varepsilon)^{-1} \psi_{i,-} dz, \quad (5.18)$$

where $\widetilde{\gamma}_1$ is a piecewise linear contour contained in the region where $\operatorname{Im} z \geq \varepsilon F_1$ and having the same endpoints as γ_1 , such that the closed contour $(-\gamma_1) \cup \widetilde{\gamma}_1$ is the positively oriented boundary of the triangle with the third corner at the point

$(E_2 + E_4)/2 + i\varepsilon F_1 + i\varepsilon^{\tilde{\delta}}$. When understanding the integral in the right hand side of (5.18), it will be convenient to decompose the function $\psi_{i,-}$ further, according to the values taken by $\operatorname{Re} P_\varepsilon$, so that we write

$$\psi_{i,-} = \psi_{i,-,2} + \psi_{i,-,4} + \tilde{\psi}_{i,-}. \quad (5.19)$$

Here the functions $\psi_{i,-,j}$ are supported in the regions where $|\operatorname{Re} P_\varepsilon - E_j| \leq \varepsilon^{3\tilde{\delta}}/\mathcal{O}(1)$, $j = 2, 4$, respectively, while the support of $\tilde{\psi}_{i,-}$ stays away from these regions.

We shall consider first the integral

$$\int_{\tilde{\gamma}_1} \left(z - \tilde{P}_\varepsilon \right)^{-1} \tilde{\psi}_{i,-} dz. \quad (5.20)$$

In the support of $\tilde{\psi}_{i,-}$, we have,

$$\left| z - \tilde{P}_\varepsilon(\rho) \right| \geq \frac{\varepsilon^{3\tilde{\delta}}}{\mathcal{O}(1)}, \quad z \in \tilde{\gamma}_1, \quad (5.21)$$

which is an essentially elliptic estimate. Using (5.12), (5.13), and (5.21) we then obtain that

$$\nabla^\ell \left((z - \tilde{P}_\varepsilon(\rho))^{-1} \tilde{\psi}_{i,-} \right) = \varepsilon^{-3\tilde{\delta}} \mathcal{O} \left(\varepsilon^{-3\tilde{\delta}\ell} \right), \quad \ell \in \mathbf{N}.$$

Also, the trace class norm of the corresponding Weyl quantization does not exceed

$$\frac{1}{h^2} \mathcal{O}(\varepsilon^{-2\tilde{\delta}}).$$

Here we may replace the function $\tilde{\psi}_{i,-}$ by another cut-off with a slightly larger support. It is therefore clear that the integral in (5.20) can be understood by constructing an h -pseudodifferential parametrix for $z - \tilde{P}_\varepsilon$, valid near the support of $\tilde{\psi}_{i,-}$, by following the same method as in the proof of Proposition 4.1. We therefore conclude that the trace integral

$$\operatorname{tr} \frac{1}{2\pi i} \int_{\tilde{\gamma}_1} \left(z - \tilde{P}_\varepsilon \right)^{-1} \tilde{\psi}_{i,-} dz$$

is equal to

$$\frac{1}{2\pi i} \frac{1}{(2\pi h)^2} \int_{\tilde{\gamma}_1} \iint \frac{1}{z - \tilde{p}_\varepsilon(\rho)} \tilde{\psi}_{i,-}(\rho) \mu(d\rho) dz + \mathcal{O} \left(\frac{h\varepsilon^{-7\tilde{\delta}}}{h^2} \right). \quad (5.22)$$

Here \tilde{p}_ε is the leading symbol of \tilde{P}_ε , where we know that $\tilde{p}_\varepsilon - p_\varepsilon = \mathcal{O}(\varepsilon)$ is supported in a region where $|\operatorname{Re} P_\varepsilon| \leq \varepsilon^{\tilde{\delta}}/\mathcal{O}(1)$, away from the set where $\operatorname{Im} P_\varepsilon \leq \varepsilon F_3 - \varepsilon\varepsilon^{\tilde{\delta}}/\mathcal{O}(1)$. According to (5.18) and (5.19), it remains to consider the integrals

$$\int_{\tilde{\gamma}_1} (z - \tilde{P}_\varepsilon)^{-1} \psi_{i,-,j} dz, \quad j = 2, 4. \quad (5.23)$$

Here an application of the bound (5.14) in Lemma 5.2 together with the fact that the trace class norm of $\psi_{i,-,j}$ on $H(\Lambda)$ for $j = 2, 4$ is

$$\mathcal{O}\left(\frac{\varepsilon^{3\tilde{\delta}}}{h^2}\right),$$

shows that the trace class norm of the expressions in (5.23) does not exceed

$$\mathcal{O}\left(\frac{\varepsilon^{3\tilde{\delta}}}{h^2}\right) \int_{\varepsilon\varepsilon^{\tilde{\delta}}}^1 \frac{1}{s} ds = \mathcal{O}(1) \frac{\varepsilon^{3\tilde{\delta}}}{h^2} \log \frac{1}{\varepsilon}. \quad (5.24)$$

Combining (5.17), (5.18), (5.19), (5.22), and (5.24) we conclude that the trace

$$\mathrm{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \psi_{i,-} dz \quad (5.25)$$

is given by

$$\frac{1}{2\pi i} \frac{1}{(2\pi h)^2} \int_{\tilde{\gamma}_1} \iint \frac{1}{z - \tilde{p}_\varepsilon(\rho)} \tilde{\psi}_{i,-}(\rho) \mu(d\rho) dz + \mathcal{O}(1) \frac{\varepsilon^{3\tilde{\delta}}}{h^2} \log \frac{1}{\varepsilon}, \quad (5.26)$$

provided that the lower bound (5.1) is strengthened to the following one,

$$h \leq \varepsilon^{10\tilde{\delta}} \log \frac{1}{\varepsilon}. \quad (5.27)$$

Here the integral

$$\int_{\tilde{\gamma}_1} \iint \frac{1}{z - \tilde{p}_\varepsilon(\rho)} \tilde{\psi}_{i,-}(\rho) \mu(d\rho) dz$$

can be replaced by the integral

$$\int_{\gamma_1} \iint \frac{1}{z - p_\varepsilon(\rho)} \psi_{i,-}(\rho) \mu(d\rho) dz,$$

at the expense of an additional error not exceeding

$$\mathcal{O}\left(\frac{\varepsilon^{3\tilde{\delta}}}{h^2}\right) \log \frac{1}{\varepsilon}.$$

The result obtained so far is typical of the behavior of the trace integrals in question in the non-elliptic region away from the tori $\widehat{\Lambda}_j$, and so we state it in the following proposition.

Proposition 5.3 Assume that $0 < \tilde{\delta} < 1$ is such that $h \leq \varepsilon^{10\tilde{\delta}} \log \frac{1}{\varepsilon}$, and recall that

$$\operatorname{Im} P_\varepsilon \leq \varepsilon F_3 - \frac{\varepsilon \varepsilon^{\tilde{\delta}}}{\mathcal{O}(1)},$$

near $\operatorname{supp}(\psi_{i,-})$, where $\psi_{i,-}$ satisfies (5.7), (5.8). We have

$$\begin{aligned} & \operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \psi_{i,-} dz \\ &= \frac{1}{2\pi i} \frac{1}{(2\pi h)^2} \int_{\gamma_1} \iint \frac{1}{z - p_\varepsilon(\rho)} \psi_{i,-}(\rho) \mu(d\rho) dz + \mathcal{O}\left(\frac{\varepsilon^{3\tilde{\delta}}}{h^2}\right) \log \frac{1}{\varepsilon}. \end{aligned} \quad (5.28)$$

It is now easy to extend the result of Proposition 5.3 to the trace integral

$$\operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} (1 - \chi_1) dz.$$

Indeed, the argument leading to the asymptotic result (5.28) remains valid also when considering the trace integrals

$$\frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \psi_{i,0} dz$$

and

$$\frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \chi_3 dz,$$

where we may also recall that $F_3 < F_1$.

When considering the expression

$$\int_{\gamma_1} (z - P_\varepsilon)^{-1} \psi_{i,+} dz,$$

we introduce a new trace class perturbation \widehat{P}_ε of P_ε , such that $\widehat{P}_\varepsilon = P_\varepsilon$ near $\operatorname{supp}(\psi_{i,+})$, and with

$$\operatorname{Im} \widehat{P}_\varepsilon \geq \varepsilon F_1 + \frac{\varepsilon \varepsilon^{\tilde{\delta}}}{\mathcal{O}(1)}$$

in the entire region where $|\operatorname{Re} P_\varepsilon| \leq \varepsilon^{\tilde{\delta}}/\mathcal{O}(1)$, and such that \widehat{P}_ε agrees with P_ε further away from this set. The natural analogue of Lemma 5.2 continues to be valid for $(z - \widehat{P}_\varepsilon)^{-1}$, and when studying the trace of

$$\int_{\gamma_1} (z - \widehat{P}_\varepsilon)^{-1} \psi_{i,+} dz,$$

we can therefore deform the contour γ_1 downwards and introduce the decomposition similar to (5.19), so that the estimate (5.21) holds for $z - \widehat{P}_\varepsilon$ along the deformed contour, near the support of $\widetilde{\psi}_{i,+}$.

The following is the main result of this subsection.

Proposition 5.4 *Assume that $E_2 < 0 < E_4$, $|E_j| \sim \varepsilon^{\widetilde{\delta}}$, $j = 2, 4$, where $0 < \widetilde{\delta} < 1$ is so small that $h \leq \varepsilon^{10\widetilde{\delta}} \log(1/\varepsilon)$. Let γ_j , $j = 1, 3$ be the horizontal segment given by $E_2 \leq \operatorname{Re} z \leq E_4$, $\operatorname{Im} z = \varepsilon F_j$. Let finally $0 \leq \chi_j \in C_0^\infty(\Lambda)$ be a cut-off function to an $\varepsilon^{\widetilde{\delta}}$ -neighborhood of $\widehat{\Lambda}_j$, $j = 1, 3$, enjoying the commutator property (5.4). We have*

$$\begin{aligned} & \operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_j} (z - P_\varepsilon)^{-1} (1 - \chi_j) dz \\ &= \frac{1}{2\pi i} \frac{1}{(2\pi h)^2} \int_{\gamma_j} \iint \frac{1}{z - p_\varepsilon(\rho)} (1 - \chi_j(\rho)) \mu(d\rho) dz + \mathcal{O}\left(\frac{\varepsilon^{3\widetilde{\delta}}}{h^2}\right) \log \frac{1}{\varepsilon}, \quad j = 1, 3. \end{aligned}$$

5.2 The Birkhoff normal form and trace integrals near the tori

In this subsection, we shall complete the trace analysis near the Diophantine levels by studying the integrals

$$\operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_j} (z - P_\varepsilon)^{-1} \chi_j dz,$$

say, when $j = 1$.

When $z \in \gamma_1$, let us consider the equation

$$(z - P_\varepsilon) u = v, \quad u \in H(\Lambda, m),$$

so that

$$(z - P_\varepsilon) \chi_1 u = \chi_1 v + [\chi_1, P_\varepsilon] u.$$

Applying the operator $U := U_1$ of Proposition 3.1, we get

$$\left(z - P_1^{(N)} - R_{N+1,1} \right) U \chi_1 u = U \chi_1 v + T_N u,$$

with the trace class norm of T_N on $H(\Lambda)$ being $\mathcal{O}(h^M)$, where M can be taken arbitrarily large, by taking the integer N in Proposition 3.1 large enough. Using the fact that $R_{N+1,1}(x, \xi, \varepsilon; h) = \mathcal{O}((h, \varepsilon, \xi)^{N+1})$ and modifying T_N slightly, we get

$$\left(z - P_1^{(N)} \right) U \chi_1 u = U \chi_1 v + T_N u.$$

It is therefore clear, in view of the fact that $\varepsilon \geq h^K$, $K \geq 1$ fixed and since the bound (5.2) holds, that

$$\chi_1 (z - P_\varepsilon)^{-1} = V \left(z - P_1^{(N)} \right)^{-1} U \chi_1 + T_{1,N}, \quad (5.29)$$

where $T_{1,N}$ has the same trace class norm bound as T_N . Here V is a microlocal inverse of U . Therefore,

$$\begin{aligned} \operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \chi_1 dz \\ = \operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_1} \left(z - P_1^{(N)} \right)^{-1} U \chi_1 V dz + \mathcal{O}(h^M). \end{aligned} \quad (5.30)$$

Here we may recall, following [16], that $U \chi_1 V = \chi(hD_x/\varepsilon^{\tilde{\delta}})$, where $\chi \in C_0^\infty(\mathbf{R}^2)$ is a standard cut-off to a neighborhood of $\xi = 0$. Let us write in what follows $\chi_\varepsilon(\xi) = \chi(\xi/\varepsilon^{\tilde{\delta}})$.

Now the eigenvalues of the translation invariant operator $P_1^{(N)}(hD_x, \varepsilon; h)$, acting on $L_\theta^2(\mathbf{T}^2)$ are given by

$$\mu(k) := P_1^{(N)} \left(h \left(k - \frac{k_0}{4} \right) - \frac{S}{2\pi} \right), \quad k \in \mathbf{Z}^2, \quad (5.31)$$

and by (5.30), we conclude that

$$\begin{aligned} \operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \chi_1 dz \\ = \sum_{k \in \mathbf{Z}^2} \frac{1}{2\pi i} \int_{\gamma_1} (z - \mu(k))^{-1} \chi_\varepsilon \left(h \left(k - \frac{k_0}{4} \right) - \frac{S}{2\pi} \right) dz + \mathcal{O}(h^M). \end{aligned} \quad (5.32)$$

We have

$$\mu(k) = \widehat{p} \left(h \left(k - \frac{k_0}{4} \right) - \frac{S}{2\pi}, \varepsilon \right) + \mathcal{O}(h), \quad (5.33)$$

where

$$\widehat{p}(\xi, \varepsilon) = p(\xi) + i\varepsilon \langle q \rangle(\xi) + \mathcal{O}(\varepsilon^2)$$

is the leading symbol of $P_1^{(N)}(hD_x, \varepsilon; h)$. We would like to compare the expression in the right hand side of (5.32) with the integral

$$\frac{1}{2\pi i} \frac{1}{(2\pi h)^2} \int_{\gamma_1} \iint (z - \widehat{p}(\xi, \varepsilon))^{-1} \chi_\varepsilon(\xi) dx d\xi dz, \quad (5.34)$$

the integration in the (x, ξ) variables being carried out over $T^*\mathbf{T}^2$.

Integrating out the x -variables in (5.34), we shall first compare the expressions

$$\int_{\mathbf{R}^2} \int_{\gamma_1} (z - \widehat{p}(\xi, \varepsilon))^{-1} \chi_\varepsilon(\xi) d\xi dz$$

and

$$\sum_{k \in \mathbf{Z}^2} h^2 \int_{\gamma_1} \left(z - \widehat{p} \left(h \left(k - \frac{k_0}{4} \right) - \frac{S}{2\pi}, \varepsilon \right) \right)^{-1} \chi_\varepsilon \left(h \left(k - \frac{k_0}{4} \right) - \frac{S}{2\pi} \right) dz.$$

When $\xi \in \mathbf{R}^2$ is such that

$$\xi \in h \left(k - \frac{k_0}{4} \right) - \frac{S}{2\pi} + [0, h)^2,$$

for some $k \in \mathbf{Z}^2$, let us write

$$[\xi] = h \left(k - \frac{k_0}{4} \right) - \frac{S}{2\pi}.$$

Let us consider

$$\int_{\mathbf{R}^2} \int_{\gamma_1} ((z - \widehat{p}(\xi, \varepsilon))^{-1} - (z - \widehat{p}([\xi], \varepsilon))^{-1}) \chi_\varepsilon(\xi) d\xi dz. \quad (5.35)$$

Let a, b be the endpoints of γ_1 . Then with suitable branches of the logarithm, we have

$$\begin{aligned} \int_{\gamma_1} ((z - \widehat{p}(\xi, \varepsilon))^{-1} - (z - \widehat{p}([\xi], \varepsilon))^{-1}) dz \\ = (\log(b - \widehat{p}(\xi, \varepsilon)) - \log(b - \widehat{p}([\xi], \varepsilon))) \\ - (\log(a - \widehat{p}(\xi, \varepsilon)) - \log(a - \widehat{p}([\xi], \varepsilon))). \end{aligned} \quad (5.36)$$

In general, for $z, w \in \mathbf{C}$, we have

$$\log z - \log w = \int_z^w \frac{1}{\zeta} d\zeta,$$

where the choice of curve joining w and z depends on the choices of branches of $\log z, \log w$. If we have the same branch then

$$|\log z - \log w| \leq \frac{C_0 |z - w|}{\min(|z|, |w|)}. \quad (5.37)$$

If the branch cut passes between z and w , we have to add a constant. In the case of (5.36), this happens precisely when $\widehat{p}(\xi, \varepsilon)$ and $\widehat{p}([\xi], \varepsilon)$ are on the opposite sides of γ_1 . Now let us concentrate on one of the terms in (5.36), say

$$\log(a - \widehat{p}(\xi, \varepsilon)) - \log(a - \widehat{p}([\xi], \varepsilon)). \quad (5.38)$$

If

$$|\operatorname{Re} a - \operatorname{Re} \widehat{p}(\xi, \varepsilon)| \leq C_0 h, \quad (5.39)$$

for a suitable fixed constant $C_0 > 0$, we estimate the two terms separately and get that the contribution to (5.35) in this case is

$$\int_{E(C_0)} \chi_\varepsilon(\xi) \log(a - \widehat{p}(\xi, \varepsilon)) d\xi \leq \mathcal{O}(1) \int_0^{C_1 h} -\log t dt = \mathcal{O}(1) h \log \frac{1}{h}.$$

Here $E(C_0) \subset \mathbf{R}^2$ is the set of all $\xi \in \mathbf{R}^2$ such that (5.39) holds. If we assume that a has been chosen so that for all $\xi \in \mathbf{R}^2$,

$$|a - \widehat{p}([\xi], \varepsilon)| \geq \frac{h}{\mathcal{O}(1)}, \quad (5.40)$$

then we get the same estimate for

$$\int \chi_\varepsilon(\xi) \log(a - \widehat{p}([\xi], \varepsilon)) d\xi.$$

In the region where $|\operatorname{Re} a - \operatorname{Re} \widehat{p}(\xi, \varepsilon)| \geq C_0 h$, let us first assume that we have the same branches of $\log(a - \widehat{p}(\xi, \varepsilon))$ and $\log(a - \widehat{p}([\xi], \varepsilon))$. Then by (5.37),

$$\log(a - \widehat{p}(\xi, \varepsilon)) - \log(a - \widehat{p}([\xi], \varepsilon)) = \mathcal{O}(1) \frac{h}{\operatorname{Re} a - \operatorname{Re} \widehat{p}(\xi, \varepsilon)},$$

and the corresponding contribution to (5.35) is

$$\mathcal{O}(h) \int_h^1 \frac{1}{t} dt = \mathcal{O}\left(h \log \frac{1}{h}\right).$$

It remains to estimate the integral of the extra contributions $\pm 2\pi i$, to $\log(a - \widehat{p}(\xi, \varepsilon)) - \log(a - \widehat{p}([\xi], \varepsilon))$ from points ξ for which $\widehat{p}(\xi, \varepsilon)$ and $\widehat{p}([\xi], \varepsilon)$ are on the opposite sides of γ_1 . But the Lebesgue measure of the set of such points is $\mathcal{O}(h)$, so the corresponding contribution to the integral is $\mathcal{O}(h)$.

Summing up our estimates and computations, we see that the expression (5.35) is $\mathcal{O}(h \log \frac{1}{h})$. Arguing similarly and using (5.33), we obtain that

$$\int_{\mathbf{R}^2} \int_{\gamma_1} ((z - \widehat{p}([\xi], \varepsilon))^{-1} - (z - \mu(k))^{-1}) \chi_\varepsilon([\xi]) d\xi dz = \mathcal{O}(h \log \frac{1}{h}). \quad (5.41)$$

Finally, we find that also,

$$\int_{\mathbf{R}^2} \int_{\gamma_1} (z - \widehat{p}([\xi], \varepsilon))^{-1} (\chi_\varepsilon(\xi) - \chi_\varepsilon([\xi])) d\xi dz = \mathcal{O}(h \log \frac{1}{h}). \quad (5.42)$$

We summarize the result of this subsection in the following proposition. Here we also use that the integral over $T^*\mathbf{T}^2$ in (5.34) can be transformed into the corresponding integral over Λ by means of the canonical transformation associated to the operator U .

Proposition 5.5 *Assume that $E_2 < 0 < E_4$, $|E_j| \sim \varepsilon^{\tilde{\delta}}$, $j = 2, 4$, where $0 < \tilde{\delta} < 1$ is so small that $h \leq \varepsilon^{10\tilde{\delta}} \log(1/\varepsilon)$. Let γ_j , $j = 1, 3$ be the horizontal segment given by $E_2 \leq \operatorname{Re} z \leq E_4$, $\operatorname{Im} z = \varepsilon F_j$. Let finally $0 \leq \chi_j \in C_0^\infty(\Lambda)$ be a cut-off function to an $\varepsilon^{\tilde{\delta}}$ -neighborhood of $\hat{\Lambda}_j$, $j = 1, 3$, enjoying the commutator property (5.4). We have*

$$\begin{aligned} \operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_j} (z - P_\varepsilon)^{-1} \chi_j dz \\ = \frac{1}{2\pi i} \frac{1}{(2\pi h)^2} \int_{\gamma_1} \iint \frac{1}{z - p_\varepsilon(\rho)} \chi_j(\rho) \mu(d\rho) dz + \mathcal{O}\left(\frac{1}{h}\right) \log \frac{1}{\varepsilon}, \quad j = 1, 3. \end{aligned}$$

5.3 End of the proof

Combining Propositions 4.2, 5.4, and 5.5, we obtain that the number of eigenvalues of P_ε in the rectangle R in (2.33) is equal to

$$\begin{aligned} \operatorname{Re} \frac{1}{2\pi i} \frac{1}{(2\pi h)^2} \int_{\gamma} \iint \frac{1}{z - p_\varepsilon(\rho)} \mu(d\rho) dz + \frac{1}{h^2} \mathcal{O}\left(\varepsilon^{3\tilde{\delta}} \log \frac{1}{\varepsilon}\right) \\ = \frac{1}{(2\pi h)^2} \iint_{\Omega(\varepsilon, R)} \mu(d\rho) + \frac{1}{h^2} \mathcal{O}\left(\varepsilon^{3\tilde{\delta}} \log \frac{1}{\varepsilon}\right). \end{aligned} \quad (5.43)$$

Here $\Omega(\varepsilon, R) = p_\varepsilon^{-1}(R) \subset \Lambda$ and $p_\varepsilon \in C^\infty(\Lambda)$ is the leading symbol of P_ε , acting on $H(\Lambda)$.

Let us now recall the C^∞ canonical transformation

$$\kappa : \operatorname{neigh}(p^{-1}(0), T^*M) \rightarrow \operatorname{neigh}(p^{-1}(0), \Lambda),$$

introduced in (3.47), so that κ is $\mathcal{O}(\varepsilon)$ -close to the identity in the C^∞ -sense, for each fixed $T \geq T_0$, $T_0 > 0$ large enough. An application of (3.48) shows that

$$p_\varepsilon(\kappa(\rho)) = p(\rho) + i\varepsilon(q - H_p G)(\rho) + \mathcal{O}(\varepsilon^2), \quad \rho \in T^*M. \quad (5.44)$$

In the compact case, we obtain the same expression for the transformed symbol.

It follows from (5.44) and the properties of the function G , recalled in Section 3, that the set $\Omega(\varepsilon, R) \subset \Lambda$ is $\mathcal{O}(\varepsilon)$ -close to the set

$$\Omega([E_2, E_4]) := \bigcup_{E_2 \leq E \leq E_4} \Omega(E) \subset T^*M,$$

introduced in (2.35). Therefore,

$$\iint_{\Omega(\varepsilon, R)} \mu(d\rho) = \iint_{\Omega([E_2, E_4])} dx d\xi + \mathcal{O}(\varepsilon). \quad (5.45)$$

Here $T > 0$ is large enough fixed. Combining (5.43) and (5.45), we complete the proof of Theorem 2.1.

6 Weyl asymptotics in the periodic case

In this section, we shall explain how the results and methods of the work [12], combined with the methods of the present work, can be used to obtain an analog of Theorem 2.1 in the case when instead of the complete integrability assumptions, we assume that Hamilton flow of p is periodic. It turns out that the analysis of the periodic case will proceed in full analogy with the previously analyzed completely integrable case. The following discussion will therefore be somewhat brief.

In order to fix the ideas, throughout this section, we shall consider the case when $M = \mathbf{R}^2$. Let P_ε be an operator satisfying all the assumptions made in subsection 2.1, and in particular, (2.12). Assume that for $E \in \text{neigh}(0, \mathbf{R})$, the following condition holds,

$$\begin{aligned} &\text{The } H_p\text{-flow is periodic on } p^{-1}(E) \cap \mathbf{R}^4 \text{ with} \\ &\text{period } T(E) > 0 \text{ depending analytically on } E. \end{aligned} \quad (6.1)$$

As in (2.21), we set

$$\langle q \rangle = \frac{1}{T(E)} \int_0^{T(E)} q \circ \exp(tH_p) dt \quad \text{on } p^{-1}(E) \cap \mathbf{R}^4, \quad (6.2)$$

and notice that the functions p and $\langle q \rangle$ are in involution, so that $H_p \langle q \rangle = 0$. Similarly to (2.24), it is established in [12] that,

$$\frac{1}{\varepsilon} \text{Im} (\text{Spec}(P_\varepsilon) \cap \{z; |\text{Re } z| \leq \delta\}) \subset \left[\min_{p^{-1}(0) \cap \mathbf{R}^4} \langle q \rangle - o(1), \max_{p^{-1}(0) \cap \mathbf{R}^4} \langle q \rangle + o(1) \right], \quad (6.3)$$

as $\varepsilon, h, \delta \rightarrow 0$.

Let

$$F_j \in \left[\min_{p^{-1}(0) \cap \mathbf{R}^4} \langle q \rangle, \max_{p^{-1}(0) \cap \mathbf{R}^4} \langle q \rangle \right], \quad j = 1, 3, \quad F_3 < F_1,$$

and let us introduce the associated level sets

$$\Lambda_{0, F_j} = \{\rho \in \mathbf{R}^4; p(\rho) = 0, \langle q \rangle(\rho) = F_j\}, \quad j = 1, 3. \quad (6.4)$$

As in [12], we shall work under the general assumption that for $j = 1, 3$,

$$T(0) \text{ is the minimal period of every } H_p\text{-trajectory in } \Lambda_{0, F_j}. \quad (6.5)$$

We shall furthermore assume that

$$dp, d\langle q \rangle \text{ are linearly independent at every point of } \Lambda_{0, F_j}, \quad j = 1, 3. \quad (6.6)$$

It follows that each connected component of the level set Λ_{0,F_j} is a two-dimensional Lagrangian torus. For simplicity, we shall assume that the sets Λ_{0,F_j} are both connected, $j = 1, 3$. We can then make a real analytic canonical transformation

$$\kappa_j : \text{neigh}(\Lambda_{0,F_j}, \mathbf{R}^4) \rightarrow \text{neigh}(\xi = 0, T^*\mathbf{T}^2),$$

given by the action-angle coordinates near Λ_{0,F_j} , so that when expressed in terms of the coordinates x and ξ on the torus side, we have $p \circ \kappa_j^{-1} = p_j(\xi_1)$, $\langle q \rangle \circ \kappa_j^{-1} = \langle q_j \rangle(\xi)$.

Following the analysis carried out in Section 4 of [12], we shall now state the following result, which can be viewed as an analog of Proposition 3.1 in the present periodic situation.

Proposition 6.1 *Let us keep all the general assumptions of subsection 2.1 and make furthermore the assumptions (6.1), (6.5), and (6.6). Assume that $\varepsilon = \mathcal{O}(h^\delta)$, $0 < \delta \leq 1$, is such $h/\varepsilon \leq \delta_0$, for some $0 < \delta_0$ sufficiently small. There exists a smooth IR-manifold $\Lambda \subset \mathbf{C}^4$ and smooth Lagrangian tori $\widehat{\Lambda}_1$ and $\widehat{\Lambda}_3 \subset \Lambda$, such that when $\rho \in \Lambda$ is away from a small but fixed neighborhood of $\widehat{\Lambda}_1 \cup \widehat{\Lambda}_3$ in Λ , we have*

$$|\text{Re } P_\varepsilon(\rho)| \geq \frac{1}{\mathcal{O}(1)} \quad (6.7)$$

or

$$|\text{Im } P_\varepsilon(\rho) - \varepsilon F_1| \geq \frac{\varepsilon}{\mathcal{O}(1)} \quad \text{and} \quad |\text{Im } P_\varepsilon(\rho) - \varepsilon F_3| \geq \frac{\varepsilon}{\mathcal{O}(1)}. \quad (6.8)$$

The manifold Λ is an $\mathcal{O}(\varepsilon + \frac{h}{\varepsilon})$ -perturbation of \mathbf{R}^4 in the C^∞ -sense, and it agrees with \mathbf{R}^4 outside of a neighborhood of $p^{-1}(0) \cap \mathbf{R}^4$. We have

$$P_\varepsilon = \mathcal{O}(1) : H(\Lambda, m) \rightarrow H(\Lambda).$$

When $j = 1, 3$, there exists an elliptic h -Fourier integral operator

$$U_j = \mathcal{O}(1) : H(\Lambda) \rightarrow L_\theta^2(\mathbf{T}^2),$$

such that microlocally near $\widehat{\Lambda}_j$, we have

$$U_j P_\varepsilon = \widehat{P}_j U_j.$$

Here $\widehat{P}_j = \widehat{P}_j(hD_x, \varepsilon, \frac{h}{\varepsilon}; h)$ is an operator acting on $L_\theta^2(\mathbf{T}^2)$ with the symbol

$$\widehat{P}_j \left(\xi, \varepsilon, \frac{h}{\varepsilon}; h \right) \sim p_j(\xi_1) + \varepsilon \sum_{k=0}^{\infty} h^k r_{j,k} \left(\xi, \varepsilon, \frac{h}{\varepsilon} \right), \quad |\xi| \leq \frac{1}{\mathcal{O}(1)},$$

where

$$r_{j,0}(\xi) = i\langle q_j \rangle(\xi) + \mathcal{O} \left(\varepsilon + \frac{h}{\varepsilon} \right),$$

and

$$r_{j,k}(\xi) = \mathcal{O} \left(\varepsilon + \frac{h}{\varepsilon} \right), \quad k \geq 1.$$

From [12], we also infer that the natural analog of Proposition 3.2 is valid, when the spectral parameter z belongs to the rectangle

$$\left[-\frac{1}{C}, \frac{1}{C}\right] + i\varepsilon \left[F_j - \frac{1}{C}, F_j + \frac{1}{C}\right], \quad j = 1, 3,$$

for a sufficiently large constant $C > 0$, and is such that $\text{dist}(z, \text{Spec}(P_\varepsilon)) \geq \varepsilon h^{N_0}$, for some $N_0 \geq 1$ fixed.

Let

$$R = (E_2, E_4) + i\varepsilon(F_3, F_1), \quad (6.9)$$

where $E_2 < 0 < E_4$ are independent of h , with $|E_j| < 1/C$, $j = 2, 4$, for $C > 0$ sufficiently large. We decompose ∂R as in (2.41), and notice that the analysis of Section 4 applies to the traces of the integrals

$$\frac{1}{2\pi i} \int_{\gamma_j} (z - P_\varepsilon)^{-1} dz, \quad j = 2, 4,$$

as it stands, so that the statement of Proposition 4.2 continues to hold in the present situation. For future reference, we state it as the following result. In its formulation, the notation $\mu(d\rho)$ stands for the symplectic volume element on Λ , and p_ε is the leading symbol of P_ε , viewed as an unbounded operator on $H(\Lambda)$.

Proposition 6.2 *Let $E_2 < 0 < E_4$ be independent of h and such that $|E_j| < 1/C$, for $C > 0$ sufficiently large, $j = 1, 2$. Assume that $\varepsilon = \mathcal{O}(h^\delta)$, $0 < \delta \leq 1$, is such that $h/\varepsilon \leq \delta_0$, for some $\delta_0 > 0$ small enough. Assume also that $0 < \tilde{\delta} < 1$ is such that $h \leq \varepsilon^{9\tilde{\delta}}$. When γ_j is the vertical segment given by $\text{Re } z = E_j$, $\varepsilon F_3 \leq \text{Im } z \leq \varepsilon F_1$, we have, for $j = 2, 4$,*

$$\begin{aligned} \text{Re tr } \frac{1}{2\pi i} \int_{\gamma_j} (z - P_\varepsilon)^{-1} dz \\ = \text{Re } \frac{1}{2\pi i} \frac{1}{(2\pi h)^2} \int_{\gamma_j} \iint \frac{1}{z - p_\varepsilon(\rho)} \mu(d\rho) dz + \frac{1}{h^2} \mathcal{O}\left(\varepsilon^{3\tilde{\delta}} \log \frac{1}{\varepsilon}\right). \end{aligned}$$

When analyzing the trace integral

$$\text{tr } \frac{1}{2\pi i} \int_{\gamma_j} (z - P_\varepsilon)^{-1} dz, \quad j = 1, 3,$$

say, with $j = 1$, we use Proposition 6.1 to introduce the following smooth partition of unity on the manifold Λ , similar to (5.3),

$$1 = \chi_1 + \chi_3 + \psi_{r,+} + \psi_{r,-} + \psi_{i,-} + \psi_{i,0} + \psi_{i,+}. \quad (6.10)$$

Here $0 \leq \chi_j \in C_0^\infty(\Lambda)$ is supported in a small flow-invariant neighborhood of $\widehat{\Lambda}_j$, where P_ε is intertwined with \widehat{P}_j , according to Proposition 6.1, and $\chi = 1$ near $\widehat{\Lambda}_j$, $j = 1, 3$. As in [13], we choose χ_j so that in the sense of trace class operators on $H(\Lambda)$, we have

$$[P_\varepsilon, \chi_j] = \mathcal{O}(h^\infty). \quad (6.11)$$

The functions $0 \leq \psi_{r,\pm} \in C_b^\infty(\Lambda)$ are such that

$$\pm \operatorname{Re} P_\varepsilon(\rho) > \frac{1}{C}, \quad (6.12)$$

near the support of $\psi_{r,\pm}$, respectively. Here, as in Section 5, we may assume that the support of $\psi_{r,+}$ is unbounded, and near infinity, the bound (6.12) improves to

$$\operatorname{Re} P_\varepsilon(\rho) \geq \frac{m(\rho)}{\mathcal{O}(1)}. \quad (6.13)$$

We now come to describe the properties of the functions $\psi_{i,\pm}$ and $\psi_{i,0}$ in (6.10). These non-negative functions in $C_0^\infty(\Lambda)$ are supported in regions invariant under the H_p -flow, and the estimate

$$\operatorname{Im} P_\varepsilon \leq \varepsilon F_3 - \frac{\varepsilon}{C}$$

holds near $\operatorname{supp} \psi_{i,-}$. Similarly, near $\operatorname{supp} \psi_{i,+}$ we have

$$\operatorname{Im} P_\varepsilon \geq \varepsilon F_1 + \frac{\varepsilon}{C},$$

and finally, the bound

$$\varepsilon F_3 + \frac{\varepsilon}{C} \leq \operatorname{Im} P_\varepsilon \leq \varepsilon F_1 - \frac{\varepsilon}{C}$$

is valid in a neighborhood of the support of $\psi_{i,0}$. As in Section 5, we may and will arrange so that in the sense of trace class operators on $H(\Lambda)$, we have

$$A[P_\varepsilon, \psi_{i,\cdot}] = \mathcal{O}(h^\infty), \quad [P_\varepsilon, \psi_{i,\cdot}]A = \mathcal{O}(h^\infty), \quad \cdot = \pm, 0. \quad (6.14)$$

Here A is a microlocal cut-off to a region where $|\operatorname{Re} P_\varepsilon| < 1/\mathcal{O}(1)$.

The analysis of the trace integrals

$$\operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \psi_{r,\pm} dz \quad (6.15)$$

proceeds exactly as in the proof of Proposition 5.1, thanks to the elliptic estimates (6.12), (6.13), and as there, we find that

$$\begin{aligned} \operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \psi_{r,\pm} dz \\ = \frac{1}{2\pi i} \frac{1}{(2\pi h)^2} \int_{\gamma_1} \iint \frac{1}{z - p_\varepsilon(\rho)} \psi_{r,\pm}(\rho) \mu(d\rho) dz + \mathcal{O}\left(\frac{1}{h}\right). \end{aligned} \quad (6.16)$$

When understanding the trace

$$\mathrm{tr} \frac{1}{2\pi i} \int_{\gamma_1} (z - P_\varepsilon)^{-1} \psi_{i,-} dz,$$

we continue to follow the analysis of Section 5, and introduce an auxiliary trace class perturbation of P_ε , concentrated in a region of the form $|\mathrm{Re} P_\varepsilon| < 1/\mathcal{O}(1)$, similar to (5.12). The arguments of Section 5 apply then as the stand, and we get the following direct analog of Proposition 5.4.

Proposition 6.3 *Assume that $E_2 < 0 < E_4$, $|E_j| < 1/\mathcal{O}(1)$, $j = 2, 4$, and let $0 < \tilde{\delta} < 1$ be so small that $h \leq \varepsilon^{12\tilde{\delta}}$. Let γ_j , $j = 1, 3$ be the horizontal segment given by $E_2 \leq \mathrm{Re} z \leq E_4$, $\mathrm{Im} z = \varepsilon F_j$. Let finally $0 \leq \chi_j \in C_0^\infty(\Lambda)$ be a cut-off function to an $\varepsilon^{\tilde{\delta}}$ -neighborhood of $\hat{\Lambda}_j$, $j = 1, 3$, enjoying the commutator property (6.11). We have*

$$\begin{aligned} & \mathrm{tr} \frac{1}{2\pi i} \int_{\gamma_j} (z - P_\varepsilon)^{-1} (1 - \chi_j) dz \\ &= \frac{1}{2\pi i} \frac{1}{(2\pi h)^2} \int_{\gamma_j} \iint \frac{1}{z - p_\varepsilon(\rho)} (1 - \chi_j(\rho)) \mu(d\rho) dz + \mathcal{O} \left(\frac{\varepsilon^{3\tilde{\delta}}}{h^2} \right) \log \frac{1}{\varepsilon}, \quad j = 1, 3. \end{aligned}$$

Here we continue to assume that $h \ll \varepsilon \leq \mathcal{O}(h^\delta)$, $\delta > 0$.

An inspection, using the normal forms near the tori $\hat{\Lambda}_j$, $j = 1, 3$, described in Proposition 6.1, shows next that the result of Proposition 5.5 remains valid in the present situation. Combining this observation with Propositions 6.2 and 6.3, we conclude that the number of eigenvalues of P_ε in the rectangle R in (6.9) is given by

$$\frac{1}{(2\pi h)^2} \iint 1_R(p_\varepsilon(\rho)) \mu(d\rho) + \frac{1}{h^2} \mathcal{O} \left(\varepsilon^{3\tilde{\delta}} \log \frac{1}{\varepsilon} \right).$$

Recalling the construction of the IR-manifold Λ , described in detail in [12], we may summarize the discussion in this section in the following result, analogous to Theorem 2.1 in the periodic case.

Theorem 6.4 *Let P_ε satisfy the general assumptions of subsection 2.1, in particular (2.12), and make further the assumption (6.1). Let*

$$F_j \in \left[\min_{p^{-1}(0) \cap \mathbf{R}^4} \langle q \rangle, \max_{p^{-1}(0) \cap \mathbf{R}^4} \langle q \rangle \right], \quad j = 1, 3, \quad F_3 < F_1,$$

be such that the assumptions (6.5) and (6.6) are satisfied. Furthermore, assume that the level sets Λ_{0,F_j} in (6.4) are connected, $j = 1, 3$. Assume that $\varepsilon = \mathcal{O}(h^\delta)$, $0 < \delta \leq 1$, is such that $h/\varepsilon \ll 1$. Let $C > 0$ be sufficiently large and let $E_2 < 0 < E_4$

be independent of h with $|E_j| < 1/C$, $j = 2, 4$. Assume finally that $\tilde{\delta} \in (0, 1)$ is so small that $h \leq \varepsilon^{12\tilde{\delta}}$. Then the number of eigenvalues of P_ε in the rectangle

$$R = (E_2, E_4) + i\varepsilon(F_3, F_1),$$

counted with the algebraic multiplicities, is equal to

$$\frac{1}{(2\pi h)^2} \iint_{E_2 \leq p \leq E_4} 1_{[F_3, F_1]}(\langle q \rangle) dx d\xi + \frac{1}{h^2} \mathcal{O}\left(\varepsilon^{3\tilde{\delta}} \log \frac{1}{\varepsilon}\right).$$

7 Complex perturbations and the damped wave equation on a surface of revolution

The purpose of this final section is to illustrate how Theorems 2.1 and 2.2 apply in the case when M is an analytic surface of revolution in \mathbf{R}^3 , and

$$P_\varepsilon = -h^2 \Delta + i\varepsilon q, \quad (7.1)$$

where Δ is the Laplace-Beltrami operator and q is an analytic function on M . When doing so, we shall restrict the attention to the same class of surfaces of revolution as in [16], [15], and begin by recalling the assumptions made on M in these previous works.

Let us normalize M so that the x_3 -axis is its axis of revolution, and parametrize it by the cylinder $[0, L] \times S^1$, $L > 0$,

$$[0, L] \times S^1 \ni (s, \theta) \mapsto (f(s) \cos \theta, f(s) \sin \theta, g(s)). \quad (7.2)$$

Here the parameter $s \in [0, L]$ is the arclength along the meridians, so that $(f'(s))^2 + (g'(s))^2 = 1$. The functions f and g are assumed to be real analytic on $[0, L]$, and we shall assume that for each $k \in \mathbf{N}$,

$$f^{(2k)}(0) = f^{(2k)}(L) = 0,$$

and that $f'(0) = 1$, $f'(L) = -1$. As we recalled in [16], these assumptions guarantee the regularity of M at the poles. We assume furthermore that M is a simple surface of revolution, in the sense that $f(s) > 0$ on $(0, L)$ has precisely one critical point $s_0 \in (0, L)$, which is a non-degenerate maximum, so that $f''(s_0) < 0$. To fix the ideas, assume that $f(s_0) = 1$. Associated to s_0 we have the equatorial geodesic $\subset M$, given by $s = s_0$, $\theta \in S^1$.

Writing

$$T^*(M \setminus \{(0, 0, g(0)), (0, 0, g(L))\}) \simeq T^*((0, L) \times S^1),$$

and using (7.2), we see that the leading symbol of $P_0 = -h^2 \Delta$ on M is given by

$$p(s, \theta, \sigma, \theta^*) = \sigma^2 + \frac{(\theta^*)^2}{f^2(s)}. \quad (7.3)$$

Here σ and θ^* are the dual variables to s and θ , respectively. Since the function p in (7.3) does not depend on θ , it follows that p and θ^* are in involution, and we recover the well-known fact that the geodesic flow on M is completely integrable.

Let $E > 0$ and $|F| < E^{1/2}$, $F \neq 0$. Then the set

$$\Lambda_{E,F} : p = E, \theta^* = F,$$

is an analytic Lagrangian torus contained inside the real energy surface $p^{-1}(E)$. Geometrically, the torus $\Lambda_{E,F}$ consists of geodesics contained between and intersecting tangentially the parallels $s_{\pm}(E, F)$ on M defined by the equation

$$f(s_{\pm}(E, F)) = \frac{|F|}{E^{1/2}}.$$

For $F = 0$, the parallels reduce to the two poles and we obtain a torus consisting of a family of meridians. The case $|F| = E^{1/2}$ is degenerate and corresponds to the equator $s = s_0$, traversed with the two different orientations. Writing $\Lambda_a := \Lambda_{1,a}$, we get a decomposition as in (2.16),

$$p^{-1}(1) \cap T^*M = \bigcup_{a \in J} \Lambda_a,$$

with $J = [-1, 1]$, $S = \{\pm 1\}$.

In [16], we have derived an explicit expression for the rotation number $\omega(\Lambda_a)$ of the torus $\Lambda_a \subset p^{-1}(1)$, $0 \neq a \in (-1, 1)$, given by

$$\omega(\Lambda_a) = \frac{a}{\pi} \int_{s_-(a)}^{s_+(a)} \frac{1}{f^2(s)} \left(1 - \frac{a^2}{f^2(s)}\right)^{-1/2} ds, \quad f(s_{\pm}(a)) = |a|. \quad (7.4)$$

We are going to assume that the analytic function $(-1, 1) \ni a \mapsto \omega(\Lambda_a)$ is not identically constant.

Let $q = q(s, \theta)$ be a real-valued analytic function on M , which we shall view as a function on T^*M . Associated to each $a \in J$, we introduce the compact interval $Q_{\infty}(\Lambda_a) \subset \mathbf{R}$ defined as in (2.22). We also define an analytic function

$$(-1, 1) \ni a \mapsto \langle q \rangle(\Lambda_a),$$

obtained by averaging q over the invariant tori Λ_a . Let us assume that the function $a \mapsto \langle q \rangle(\Lambda_a)$ is not identically constant.

Example. Assume that $q = q(s)$ depends on s only. Then it was shown in [16] that for all $0 \neq a \in (-1, 1)$, we have

$$Q_{\infty}(\Lambda_a) = \{\langle q \rangle(\Lambda_a)\},$$

where

$$\langle q \rangle(\Lambda_a) = \frac{J(q, a)}{J(1, a)}, \quad J(\psi, a) = \int_{s_-(a)}^{s_+(a)} \psi(s) \frac{f(s)}{(f^2(s) - a^2)^{1/2}} ds, \quad f(s_{\pm}(a)) = |a|.$$

Generalizing the example above, an analytic family of the form

$$q_\eta(s, \theta) = q_0(s) + \eta q_1(s, \theta), \quad q_0 = q, \quad 0 < \eta \ll 1,$$

was also considered in [16], and it was shown that the set

$$\bigcup_{\Lambda_a \in \omega^{-1}(\mathbf{Q}) \cup S} Q_{\infty, \eta}(\Lambda_a), \quad S = \{\pm 1\},$$

has a small measure compared with that of $\{\langle q_\eta \rangle(\Lambda_a); a \in (-1, 1)\}$, and that consequently, there exists a rich set of values $F \in \bigcup_{a \in J} Q_{\infty, \eta}(\Lambda_a)$ satisfying the assumptions (2.27), (2.28), (2.29), and (2.36) for q_η . Here $Q_{\infty, \eta}(\Lambda_a)$ is the range of the limit of $\langle q_\eta \rangle_T$, as $T \rightarrow \infty$, along Λ_a .

The following result is a consequence of Theorem 2.1.

Proposition 7.1 *Assume that $M \subset \mathbf{R}^3$ is a simple analytic surface of revolution given by the parametrization (7.2), for which the rotation number $\omega(\Lambda)$, $\Lambda = \Lambda_a$, defined in (7.4), is not identically constant. Consider an operator of the form $P_\varepsilon = -h^2 \Delta + i\varepsilon q$, with $q = q_\eta$ as above, such that the analytic function $(-1, 1) \ni a \mapsto \langle q \rangle(\Lambda)$, given by the torus averages of q , is not identically constant and extends continuously to $[-1, 1]$. There exists a subset $\mathcal{E}_\eta \subset \bigcup_{\Lambda \in J} Q_\infty(\Lambda)$ of measure $\mu(\eta)$ tending to zero with η , such that the conclusion of Theorem 2.1 holds uniformly for $F_j \in \bigcup_{\Lambda \in J} Q_\infty(\Lambda) \setminus \mathcal{E}_\eta$, and gives the number of eigenvalues of P_ε in the region*

$$[E_2, E_4] + i\varepsilon [F_3, F_1], \quad E_2 < 1 < E_4, \quad |E_j - 1| \sim \varepsilon^{\tilde{\delta}}.$$

Remark. A more precise description of the complement of the set \mathcal{E}_η in Proposition 7.1, given in terms of (α, β, γ) -good values, can be found in Section 7 of [16].

Remark. Let us remark finally that when discussing the operator P_ε given by (7.1), it would have been possible to allow the analytic function q on M to depend holomorphically on the spectral parameter $z \in \text{neigh}(1, \mathbf{C})$, with q real-valued for z real — see also the discussion in Section 6 of [13] for a similar observation in the periodic case. Such an extension is motivated by the problem of studying spectral asymptotics for the damped wave equation with an analytic damping coefficient, considered on the analytic surface of revolution M .

We shall now apply this remark to the situation of the damped wave equation

$$(-\Delta + 2a(x)\partial_t + \partial_t^2)v(t, x) = 0, \quad \text{on } \mathbf{R} \times M,$$

where $a(x)$ is analytic and real-valued. An important role is played here by the stationary solutions $e^{i\tau t}u(x)$, $u \not\equiv 0$ and the corresponding eigenfrequencies $\tau \in \mathbf{C}$, given by the equation

$$(-\Delta + 2ia(x)\tau - \tau^2)u(x) = 0, \quad (7.5)$$

and we are interested in the asymptotic distribution of the eigenfrequencies τ . We know that the large eigenfrequencies are confined to a band along the real axis (see [20], [33] for such results under more general assumptions), and that the real parts obey the same Weyl law as for the corresponding Helmholtz equation $(-\Delta - \tau^2)u = 0$. Less is known about the asymptotic distribution of the imaginary parts, and here we can apply Proposition 7.1 and the subsequent remark.

The set of eigenfrequencies is symmetric under reflection in the imaginary axis, so we can concentrate on the case when $\operatorname{Re} \tau \gg 1$. We make a semiclassical reduction by putting $\tau = w/h$, $0 < h \ll 1$, $\operatorname{Re} w \sim 1$ and get

$$(-h^2\Delta + 2ihwa(x) - w^2)u = 0, \quad (7.6)$$

and we can apply our results with $z = w^2$, $\varepsilon = h$, $q = 2\sqrt{z}a$, and with $P = -h^2\Delta$ as the unperturbed operator with leading symbol $p(x, \xi) = \xi^2$. It may now also be useful to recall the following general bounds on the imaginary part of an eigenfrequency w in (7.6),

$$h \left(\lim_{T \rightarrow \infty} \inf_{p^{-1}(1)} \langle a \rangle_T - o(1) \right) \leq \operatorname{Im} w \leq h \left(\lim_{T \rightarrow \infty} \sup_{p^{-1}(1)} \langle a \rangle_T + o(1) \right).$$

This result was obtained in [20] — see also (2.24) for the present completely integrable case.

The function $\langle a \rangle_\infty = \lim_{T \rightarrow \infty} \langle a \rangle_T$ is homogeneous of degree 0 in ξ , thanks to the homogeneity properties of the H_p -flow. This means that the set $E_2 \leq \operatorname{Re} w \leq E_4$, $hF_3 \leq \operatorname{Im} w \leq hF_1$ corresponds to the set $\{(x, \xi) \in T^*M; E_2^2 \leq \xi^2 \leq E_4^2, F_3 \leq \langle a \rangle_\infty \leq F_1\}$ and we notice that the conditions imposed on F_j (i.e. on the properties of the corresponding torus, where $p = E$, $\langle a \rangle_\infty = F_j$) in Proposition 7.1 in the case when $q = a$, are independent of the real energy $p = E$. Applying Proposition 7.1 and the subsequent remark, we get

Theorem 7.2 *Consider the stationary damped wave equation in the equivalent forms (7.5) and (7.6). Assume that the assumptions of Proposition 7.1 are fulfilled with $q = a$. Then uniformly for $F_1, F_3 \in \bigcup_{\Lambda \in J} Q_\infty(\Lambda) \setminus \mathcal{E}_\eta$, the number of eigenfrequencies w of (7.6) in the region*

$$[E_2, E_4] + ih[F_3, F_1], \quad E_2 < 1 < E_4, \quad |E_j - 1| \sim h^{\tilde{\delta}},$$

is equal to $(2\pi h)^{-2}$ times

$$\operatorname{vol} \left\{ (x, \xi) \in T^*M; E_2^2 \leq \xi^2 \leq E_4^2, Q_\infty(\Lambda \left(x, \frac{\xi}{|\xi|} \right)) \subset [F_3, F_1] \right\} + \mathcal{O}(h^{3\tilde{\delta}} \ln \frac{1}{h}).$$

More generally, the number of eigenfrequencies w of (7.6) in the region

$$[E_2, E_4] + ih[F_3, F_1], \quad E_2 < 1 < E_4, \quad |E_j| \sim 1,$$

is equal to

$$\frac{1}{(2\pi h)^2} \left(\text{vol} \left\{ (x, \xi) \in T^*M; E_2^2 \leq \xi^2 \leq E_4^2, Q_\infty(\Lambda(x, \frac{\xi}{|\xi|})) \subset [F_3, F_1] \right\} \right) + \mathcal{O}(h^{2\tilde{\delta}-2}).$$

The number of eigenfrequencies τ of (7.5) in the region $[E_2, E_4] + i[F_3, F_1]$, where $E_2 \leq E_4$, $E_j \sim 1/h \gg 1$, is equal to

$$\frac{1}{(2\pi)^2} \left(\text{vol} \left\{ (x, \xi) \in T^*M; E_2^2 \leq \xi^2 \leq E_4^2, Q_\infty(\Lambda\left(x, \frac{\xi}{|\xi|}\right)) \subset [F_3, F_1] \right\} \right) + \mathcal{O}(h^{2\tilde{\delta}-2}).$$

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